Synthesis of Chebyshev Four-Bar Linkages and Analysis of Their Motion Properties via the Centrodes

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Abstract. The subject of this paper is the synthesis of Chebyshev four-bar linkages as three-poses rigid body guidance for obtaining an approximated long straight path of the coupler link midpoint. Moreover, the kinematic analysis of each type of the designed Chebyshev four-bar linkages is formulated in order to validate the proposed design procedure and to analyze the coupler rigid body motion by means of the centrodes, which take the form of approximated, circle and straight line (cycloidal motion) or internal circles (hypocycloidal motion) that tend to Cardan circles at infinity. Finally, applying the Roberts-Chebyshev theorem, the corresponding six-bar mechanisms are designed to obtain approximated long straight path parallel motions. Thus, implementing this formulation in Matlab, the proposed design procedure has been validated.

Keywords: Chebyshev mechanism · synthesis · analysis · centrodes · parallel motion

1 Introduction

Planar linkages are among the most widely used mechanisms, especially in the field of automatic machines, in order to obtain trajectories of interest. The kinematic synthesis of these mechanisms is typically aimed at the design of the coupler curve of the mechanism itself [1, 2]. Among the most used trajectories are straight-line paths, both exact and approximate, and the planar mechanisms capable of realizing them have been extensively studied in the field of machine theory [3]. In some applications, the synthesis of parallel motion generator mechanisms [4, 5] may be necessary. The applications of linkages in the field of robotics and automation are the most varied as reported in [6]. The synthesis methodologies are based either on the use of rigid body guidance [7], but also on the use of geometric loci, such as the inflection circle [8]. The inflection circle can also be used as an analysis tool [9, 10]. Together with this, the centrodes can also be an important analysis tool, which can provide the designer with important information in the synthesis phase [11–13].

This paper deals with the synthesis of Chebyshev four-bar linkages as three-poses rigid body guidance for obtaining an approximated long straight path of the coupler link midpoint. Moreover, the kinematic analysis formulation, along with the proposed design procedure, have been implemented in Matlab and validated.

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2 Chebyshev Mechanism: Synthesis and Analysis

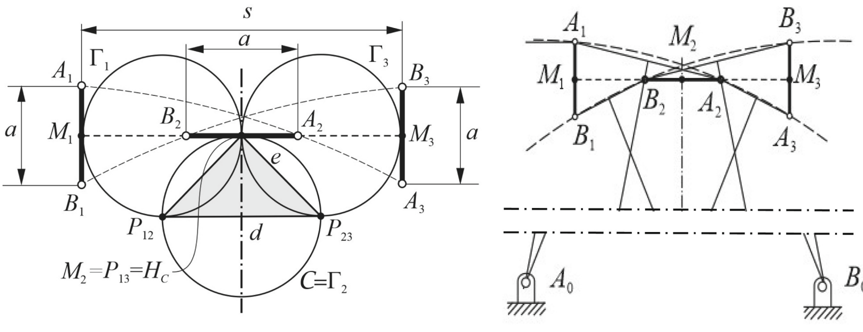
Referring to Fig. 1a, a general Chebyshev mechanism can be synthesized in the form of a cross four-bar linkage by assigning at distance *s*, the three double-symmetric poses A_1B_1, A_2B_2 and A_3B_3 with the corresponding aligned midpoints M_1, M_2 and M_3 of the coupler link of *a* length. In particular, the right pole triangle of vertices P_{12}, P_{23} and P_{31} is obtained by considering the three corresponding finite rotations, between the poses 1 and 2, 2 and 3, along with that between 1 and 3 and consequently, the circumcircle *C* is traced along with the circles Γ_1, Γ_2 and Γ_3, which are derived as mirror of *C* with respect to the sides P_{12}-P_{13}, P_{12}-P_{23} and P_{13}-P_{23} of the right pole triangle, respectively. The orthocenter H_C coincides with P_{13} and Γ_2 is coincident with *C* since P_{12}-P_{23} is the diameter of length $d = s/2$ by which the cathetus length $e = d/\sqrt{2}$. Thus, the Chebyshev four-bar linkage is designed by assuming *A* and *B* as circling points and determining the corresponding center points A_0 and B_0 by intersecting the perpendicular bisectors to both segments A_1A_2 and A_2A_3 for A_0 and to B_1B_2 and B_2B_3 for B_0, as shown in Fig. 1b.

The proposed mechanism design procedure involves three cases, which are obtained for $s = 2a$, $s > 2a$ and $s < 2a$ respectively, and thus comparing the distance *s* between the coupler link midpoints M_1 and M_3, with the length *a* of the coupler link *AB*.

The first two cases are of particular interest, since they give an approximated and quite long straight path between points M_1 and M_3, while the case for $s < 2a$ gives a midpoint curvilinear path that still passes through the points M_1, M_2 and M_3 because of the initial design specifications. Thus, only the first two cases are considered in the following.

Moreover, the kinematic analysis of each type of the designed Chebyshev four-bar linkages is formulated in order to validate the proposed design procedure and to analyze the coupler rigid body motion by means of the centrodes and inflection circle.

Thus, referring to Fig. 2a and assuming the fixed frame *OXY* with the *X*-axis passing through the fixed revolute joints A_0 and B_0, the position analysis is formulated by choosing the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and $\mathbf{r}_4$ along the corresponding links of the four-bar linkage.



a) b)

Fig. 1. The proposed design procedure a) three-poses with aligned midpoints; b) synthesis of the Chebyshev four-bar linkage (the sketch is horizontally cut since too high).

In particular, the following loop-closure equation is obtained

$$\mathbf{r}_2 + \mathbf{r}_3 = \mathbf{r}_1 + \mathbf{r}_4 \tag{1}$$

where each vector $\mathbf{r}_i$ for $i = 1, \ldots, 4$ can be expressed in matrix form as follows

$$\mathbf{r}_i = \left[r_i \cos\theta_i,\ r_i \sin\theta_i,\ 1 \right]^T \tag{2}$$

where r_i and θ_i are the magnitude and the counterclockwise angle of vector $\mathbf{r}_i$ respectively, while T indicates the transpose.

Developing Eq. (1), one has

$$\theta_4 = 2\tan^{-1}\frac{-\mathrm{B} + \sigma\sqrt{\mathrm{B}^2 - \mathrm{C}^2 + \mathrm{A}^2}}{\mathrm{C} - \mathrm{A}} \tag{3}$$

where σ is equal to ±1 according to the assembly mode and the coefficients A, B and C are expressed as function of the driving angle θ_2 by

$$\begin{aligned} \mathrm{A} &= 2\, r_2 (r_1 - r_4 \cos\theta_2) \\ \mathrm{B} &= -2\, r_2 r_4 \sin\theta_2 \\ \mathrm{C} &= r\_1^2 + r\_2^2 + r\_4^2 - r\_3^2 - 2\, r_1\, r_2 \cos\theta_2 \end{aligned} \tag{4}$$

and

$$\theta_3 = \tan^{-1}\frac{r_4 \sin\theta_4 - r_2 \sin\theta_2}{r_1 + r_4 \cos\theta_4 - r_2 \cos\theta_2} \tag{5}$$

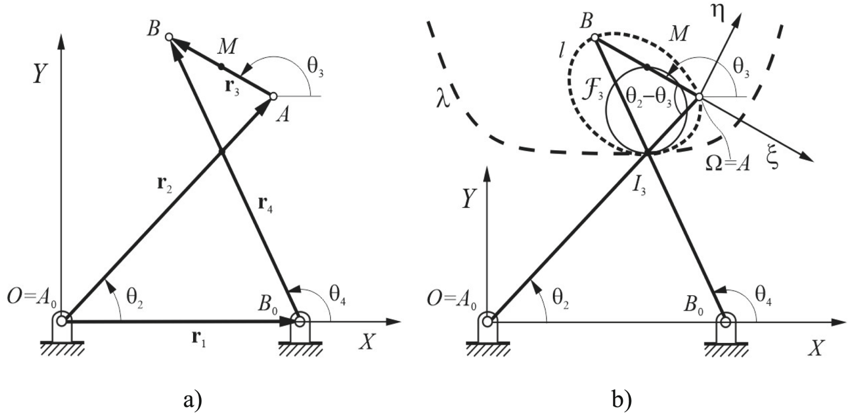


Fig. 2. Chebyshev mechanism: a) vectors loop; b) centrodes and inflection circle.

Referring to Fig. 2b and according to the Aronhold-Kennedy theorem, the fixed centrode λ_3 of the coupler link 3 (AB) can be expressed in vector form as follows

$$\mathbf{r}_{\lambda 3} = \mathbf{r}_{P3} = \left[\frac{x_{A0} y_A (x_{B0} - x_B) + x_{B0} y_B (x_A - x_{A0})}{y_B (x_A - x_{A0}) + y_A (x_{B0} - x_B)},\ \frac{y_A y_B (x_{A0} - x_{B0})}{y_B (x_A - x_{A0}) + y_A (x_{B0} - x_B)},\ 1 \right]^T \tag{6}$$

where $\mathbf{r}_{P3}$ is the position vector of the instantaneous center of rotation P_3 of 3 and the x and y Cartesian coordinates of points A_0, B_0, A and B corresponds to the components of vectors $\mathbf{r}_i$ for $i = 1, \ldots, 4$ as given by Eq. (2).

Similarly, the moving centrode l_3 is expressed as

$$\mathbf{r}_{l3} = \frac{r_3 \sin(\theta_4 - \theta_3)}{\sin(\theta_4 - \theta_2)} \left[\cos(\theta_2 - \theta_3),\ \sin(\theta_2 - \theta_3),\ 1\right]^T \tag{7}$$

where $\mathbf{r}_{l3}$ is the position vector of P_3 with respect to moving frame $\Omega\xi\eta$ that is attached to the coupler link 3, while the oriented angles θ_3 and θ_4 are given by Eqs. (5) and (3) respectively, since θ_2 is the driving angle and $r_3 = AB = a$.

Moreover, in order to simulate the pure-rolling motion of the moving centrode l_3 on the fixed one λ_3, the position vector $\mathbf{r}_{l3}$ of P_3 must be referred to the fixed frame OXY and this can be done by multiplying it for the following transformation matrix $\mathbf{T}_3$:

$$\mathbf{T}_3 = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & x_\Omega \\ \sin\theta_3 & \cos\theta_3 & y_\Omega \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

which takes into account of the coupler motion with respect to OXY by means of the oriented angle θ_3 and the Cartesian coordinates x_Ω and y_Ω of the origin Ω.

Given that the Chebyshev four-bar linkage is aimed to generate an approximated straight path of the coupler midpoint M, the inflection circle I_3 is of great aid to check the geometric performance of the designed mechanism.

Therefore, this geometric locus of all coupler points which show an inflection point in their paths and that is always tangent to both centrodes at the instantaneous center of rotation P_3, is determined by using the well-known Euler-Savary equation, as follows

$$(P_3A)^2 = A_0A \cdot A'A \quad A'A = \frac{(P_3A)^2}{A_0A} \tag{9}$$

$$(P_3B)^2 = B_0B \cdot B'B \tag{10}$$

where A' and B' are the corresponding points of A and B that lie on I_3 and $A'A$ and $B'B$ are oriented segments that are positive from A' and B' to A and B, respectively.

Consequently, the equation of the inflection circle I_3 is obtained by applying the analytical geometry from the knowledge of the three points P_3, A' and B'.

3 Chebyshev Six-Bar Parallel Motion Generator

Referring to Fig. 3a and applying the Roberts-Chebyshev theorem, the two cognates crank-rocker lambda mechanisms A_0CEE_0 and B_0DFE_0 are obtained by the well-known graphical construction from the double-rocker Grashof four-bar linkage A_0ABB_0.

Thus, referring to Fig. 3b and combining the cognates crank-rocker lambda mechanisms by joining rigidly both cranks A_0C and B_0D and also the coupler points M_I and M_II with a novel link, a one time redundant seven-bar mechanism is obtained, by which, eliminating the redundant link E_0F (dashed line), the final crank-driven Chebyshev six-bar parallel motion generator of Fig. 3c is achieved.

The main kinematic characteristics of this six-bar mechanism are that it is crank-driven, the rigid body motion of link M_IM_{II} is a pure translation (parallel motion) and each point shows an approximated straight path according to the Chebyshev four-bar linkage.

Referring to Fig. 4 and assuming the fixed frame *OXY* with the *X*-axis passing through the fixed revolute joints A_0 and E_0, the position analysis is formulated by choosing the vectors $\mathbf{r}_1$, $\mathbf{r}_{2C}$, $\mathbf{r}_{2D}$, $\mathbf{r}_{3E}$, $\mathbf{r}_{3M}$, $\mathbf{r}_4$, $\mathbf{r}_5$ and $\mathbf{r}_6$ along the corresponding links of the six-bar linkage. In particular, the following loop-closure equations are obtained

$$\mathbf{r}_{2C} + \mathbf{r}_{3E} = \mathbf{r}_1 + \mathbf{r}_5 \tag{11}$$

$$\mathbf{r}_{2D} + \mathbf{r}_4 + \mathbf{r}_6 = \mathbf{r}_1 + \mathbf{r}_5 + \mathbf{r}_{3M} \tag{12}$$

where all vectors can be expressed in the same form of Eq. (2).

Developing Eq. (11), the oriented angles θ_3 and θ_5 are given by

$$\theta_3 = \tan^{-1}\frac{r_5\sin\theta_5 - r_{2C}\sin(\theta_2+\pi)}{r_1 + r_5\cos\theta_5 - r_{2C}\cos(\theta_2+\pi)} \tag{13}$$

$$\theta_5 = 2\tan^{-1}\frac{-\mathrm{E} + \sigma\sqrt{\mathrm{E}^2 - \mathrm{F}^2 + \mathrm{E}^2}}{\mathrm{F} - \mathrm{D}} \tag{14}$$

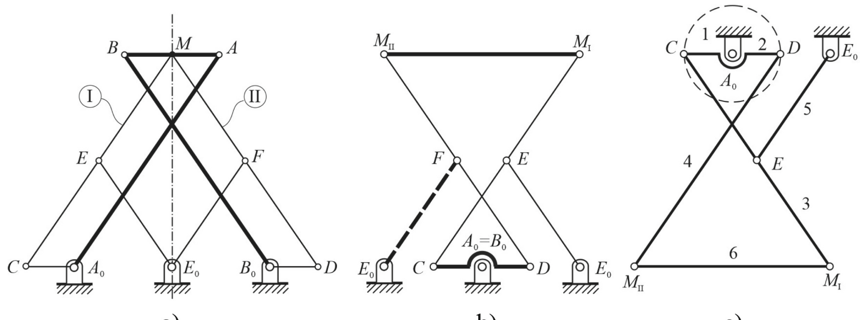


Fig. 3. Graphical synthesis: a) application of Roberts-Chebyshev theorem; b) synthesis by combining two cognate linkages: c) crank-driven Chebyshev six-bar parallel motion generator.

where σ is equal to ± 1 according to the assembly mode and the coefficients *D*, *E* and *F* are expressed as function of the driving crank-angle θ_2 by

$$\begin{aligned} \mathrm{D} &= 2\, r_5[r_1 - r_{2C}\cos(\theta_2+\pi)] \\ \mathrm{E} &= 2\, r_5 - r_{2C}\sin(\theta_2+\pi) \\ \mathrm{F} &= r\_1^2 + r\_2C^2 + r\_5^2 - r\_3E^2 + 2\, r_1\, r_{2C}\cos\theta_2 \end{aligned} \tag{15}$$

Developing the loop-closure Eq. (12) and referring to Fig. 4, the oriented angle θ_6 is equal to 180° because it remains always parallel to the fixed frame 1 (parallel motion), while θ_4 can be expressed as follows

$$\theta_4 = \tan^{-1} \frac{r_5 \sin\theta_5 + r_3 \sin\theta_{3M} - r_{2D} \sin\theta_2 - r_6 \sin\theta_6}{r_1 + r_5 \cos\theta_5 + r_3 \cos\theta_{3M} - r_{2D} \cos\theta_2 - r_6 \cos\theta_6} \quad (16)$$

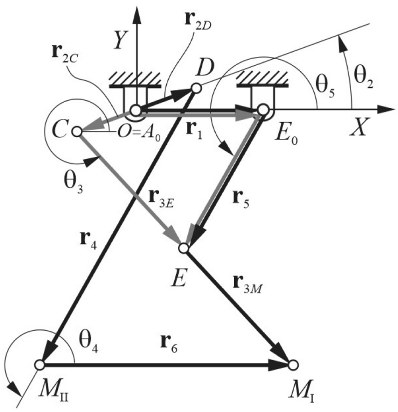


Fig. 4. Chebyshev six-bar parallel motion generator: vectors loop.

4 Graphical and Numerical Results

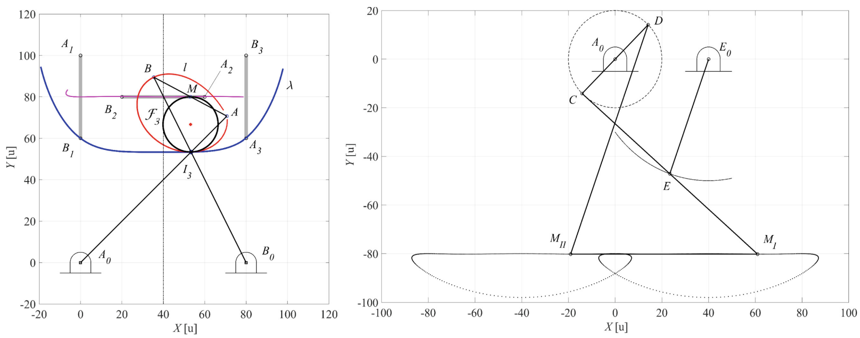
The proposed formulation regarding the synthesis and the analysis of the Chebyshev four-bar linkage and then the Chebyshev six-bar parallel motion generator, has been implemented in Matlab to validate the proposed design procedure.

In particular, as approximated straight lines generators, both cases for $s = 2a$ and $s > 2a$ have been considered, in order to validate the proposed design procedure with the aid of the fixed and moving centrodes and also the inflection circle.

Thus, referring first to the example of Fig. 5a, the Chebyshev four-bar linkage is represented along with the centrodes, the inflection circle, the approximated straight path and the initial three poses. In particular, the input data are: $\theta_2 = 45°$, $r_1 = 80$ u, $r_2 = r_4 = 40$ u, $r_3 = 40$ u, $A_1 = (0, 100)$ u, $B_1 = (0, 60)$ u, $A_2 = (60, 80)$ u, $B_2 = (20, 80)$ u, $A_3 = (0, 100)$ u, $B_3 = (80, 100)$ u, which point coordinates give the three poses for $s = 2a$.

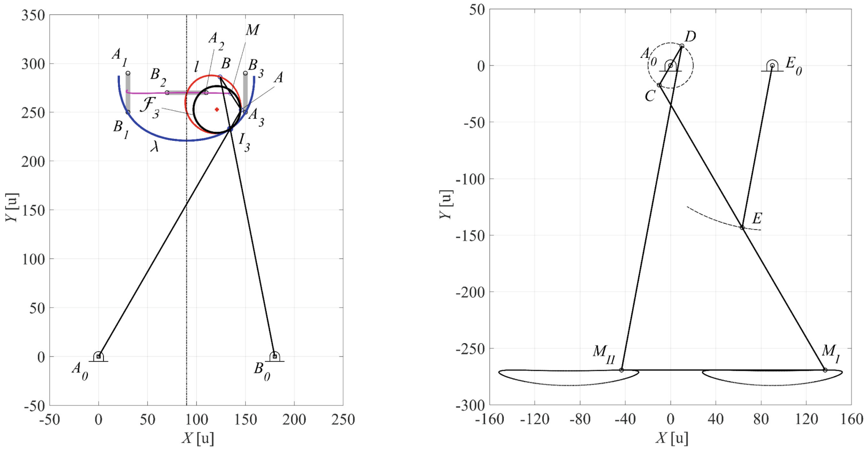
Likewise, Fig. 5b shows the Chebyshev six-bar parallel motion generator for $\theta_2 = 45°$, $r_1 = 40$ u, $r_{2D} = r_{2C} = 20$ u, $r_{3E} = r_{3M} = 50$ u, $r_4 = 100$ u, $r_5 = 50$ u, $r_6 = 80$ u.

The second example is shown in Fig. 6a for $\theta_2 = 60°$, $r_1 = 180$ u, $r_2 = r_4 = 291.5$ u, $r_3 = 40$ u, $A_1 = (30, 290)$ u, $B_1 = (30, 250)$ u, $A_2 = (110, 270)$ u, $B_2 = (70, 270)$ u, $A_3 = (150, 250)$ u, $B_3 = (150, 290)$ u, which point coordinates give the three poses for $s > 2a$. Likewise, Fig. 6b shows the Chebyshev six-bar parallel motion generator for $\theta_2 = 60°$, $r_1 = 90$ u, $r_{2D} = r_{2C} = 20$ u, $r_{3E} = r_{3M} = 145.75$ u, $r_4 = 291.5$ u, $r_5 = 145.75$ u, $r_6 = 180$ u.



a) b)

Fig. 5. Case for $s = 2a$ with $\theta_2 = 45°$: a) four-bar linkage along with the approximated cycloidal centrodes and inflection circle; b) six-bar parallel motion generator.



a) b)

Fig. 6. Case for $s > 2a$ with $\theta_2 = 60°$: a) four-bar linkage, along with the approximated hypocycloidal centrodes and inflection circle; b) six-bar parallel motion generator.

5 Conclusions

The synthesis of Chebyshev four-bar linkages has been proposed as three-poses rigid body guidance with aligned coupler midpoints and double symmetric poses, in order to obtain approximated long straight paths of the coupler link midpoint. Comparing the maximum distance between the coupler midpoint positions with the coupler length, two main cases have been considered as straight path generators, which approximate the cycloidal and the hypocycloidal rigid body motions, respectively. Moreover, the kinematic analysis formulation, along with the proposed design procedure, have been implemented in Matlab and validated by means of significant examples.

Finally, applying the Roberts-Chebyshev theorem, the corresponding Chebyshev six-bar parallel generators have been designed and validated for both design cases.

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