Primality of multiply connected polyominoes

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Abstract It is known that the polyomino ideal of simple polyominoes is prime. In this paper, we focus on multiply connected polyominoes, namely polyominoes with holes, and observe that the nonexistence of a certain sequence of inner intervals of the polyomino, called zig-zag walk, gives a necessary condition for the primality of the polyomino ideal. Moreover, by computational approach, we prove that for all polyominoes with rank less than or equal to 14, the above condition is also sufficient. Lastly, we present an infinite new class of prime polyomino ideals.

1. Introduction

Polyominoes are two-dimensional objects obtained by joining edge by edge squares of the same size. They are studied from the point of view of combinatorics (e.g., in tiling problems of the plane), as well as from the point of view of commutative algebra (e.g., associating binomial ideals to polyominoes). The latter were introduced by Qureshi in [7]. In particular, she introduces a binomial ideal generated by the inner 2-minors of a polyomino, called a polyomino ideal. We refer the reader to Section 2 for the notation.

Two pending and questions of interest regarding polyomino ideals are to classify those that are prime and to prove if they are radical ideals. In this work, we focus on the first question, giving a partial answer in terms of their geometric realization. Briefly, a polyomino is called prime if its polyomino ideal is prime. In [3, 4], and [8], the authors prove that a polyomino is prime if and only if it is balanced, and that the simple polyominoes are prime. A simple polyomino is a polyomino without holes, whereas polyominoes having one or more holes are called multiply connected polyominoes, using the terminology adopted in [1], an introductory book on polyominoes.

In general, giving a complete characterization of the primality of multiply connected polyomino ideals is not so easy. A family of prime polyominoes obtained by removing a convex polyomino by a given rectangle was shown in [5] and [9].

In Section 3, we give a necessary condition for the primality of the polyomino ideal with respect to the geometric representation of the polyomino. This condition is related to a sequence of inner intervals contained in the polyomino, called a zig-zag walk (see Definition 3.2), whose existence determines the nonprimality of the polyomino ideal.

It is known that a polyomino ideal that is prime is a toric ideal. We present a toric ideal associated to a polyomino, generalizing Shikama's construction in [9]. This toric

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ideal contains the polyomino ideal (see Proposition 3.1). Moreover, if the polyomino contains a zig-zag walk, the binomial associated to the zig-zag walk belongs to the toric ideal, and the above inclusion is strict.

The condition on zig-zag walks gives us a good filtration of primality. In fact, as an application, by implementing the algorithm described in [6], we compute all the polyominoes with rank less than or equal to 14 that are 123851 (for a complete description of the algorithm, see [6]). By the computational approach, using `Macaulay2` [2], we obtain the following theorem.

THEOREM 1.1
Let $\mathcal{P}$ be a polyomino with $\operatorname{rank}(\mathcal{P}) \leq 14$. *The following conditions are equivalent:*

(1) *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
(2) *$\mathcal{P}$ contains no zig-zag walk.*

In the final section of this paper, we observe that removing 5 squares in a particular position from a given rectangle, we obtain a polyomino with a zig-zag walk (see Figure 6(B)). Moreover, we define a new infinite family of polyominoes that we call grid polyominoes, which are obtained by removing rectangular holes by a given rectangle in a way that avoids the existence of zig-zag walks. We prove that grid polyominoes are primes.

Therefore, the natural conjecture arises as follows.

CONJECTURE 1.2
Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:

(1) *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
(2) *$\mathcal{P}$ contains no zig-zag walks.*

2. Preliminaries

In this section, we recall definitions and notation first introduced by Qureshi in [7]. Let $a = (i, j), b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$; the set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an *interval* of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$. In particular, a, b are called *diagonal corners* and c, d *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the left lower (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a + (1, 0), a + (0, 1), a + (1, 1)$ and its edges $E(C)$ are

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}.$$

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected* if there is a sequence of cells $C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. In addition, if $C_i \neq C_j$

for all $i \neq j$, then $C_1, \dots, C_m$ is called a *path* (connecting C and D). A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P}) = \bigcup_{C \in \mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and we denote it by $\operatorname{rank}(\mathcal{P})$.

A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$. We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \dots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \dots, m$. If the polyomino is not simple, then it is multiply connected (see [1]).

A finite collection $\mathcal{H}$ of cells not in $\mathcal{P}$ is called a *hole* of $\mathcal{P}$ if any two cells in $\mathcal{H}$ are connected through a path of cells in $\mathcal{H}$, and $\mathcal{H}$ is maximal with respect to the inclusion. Note that a hole $\mathcal{H}$ of a polyomino $\mathcal{P}$ is itself a simple polyomino.

Following [4], an interval $[a, b]$ with $a = (i, j)$ and $b = (k, \ell)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = \ell$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \dots, k - 1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it *maximal horizontal edge interval*. Similarly, one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$.

Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in V(\mathcal{P})$. We define on the vertices of $\mathcal{P}$ the following total order: $a < b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$.

Let $\mathcal{P}$ be a polyomino, and let $\mathbb{K}$ be a field. We denote by S the polynomial over $\mathbb{K}$ with variables x_v, where $v \in V(\mathcal{P})$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where c, d are the anti-diagonal corners of $[a, b]$. We denote by $\mathcal{M}$ the set of all inner 2-minors of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset S$ generated by $\mathcal{M}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $\mathbb{K}[\mathcal{P}] = S / I_{\mathcal{P}}$.

3. The toric ring of generic polyominoes and zig-zag walks

Let $\mathcal{P}$ be a polyomino. Let $S = \mathbb{K}[x_v | v \in V(\mathcal{P})]$ and $I_{\mathcal{P}} \subset S$ the polyomino ideal associated to $\mathcal{P}$. Let $\mathcal{H}$ be a hole of $\mathcal{P}$. The minimum, with respect to $<$, of the vertices of $\mathcal{H}$ is called the *lower left corner* of $\mathcal{H}$.

Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be holes of $\mathcal{P}$. For $k = 1, \dots, r$, we denote by $e_k = (i_k, j_k)$ the lower left corner of $\mathcal{H}_k$. For $k \in K = \{1, \dots, r\}$, we define the following subset of $V(\mathcal{P})$:

$$\mathcal{F}_k = \big\{(i, j) \in V(\mathcal{P}) \mid i \leq i_k \text{ and } j \leq j_k\big\}.$$

Let $\{V_i\}_{i \in I}$ be the set of all the maximal vertical edge intervals of $\mathcal{P}$, and $\{H_j\}_{j \in J}$ be the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i \in I}, \{h_j\}_{j \in J}$, and $\{w_k\}_{w \in K}$ be three sets of variables associated to $\{V_i\}_{i \in I}, \{H_j\}_{j \in J}$, and $\{\mathcal{F}_k\}_{k \in K}$, respectively. We consider the map:

$$\alpha : V(\mathcal{P}) \longrightarrow \mathbb{K}\big[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K\big]$$
$$a \longmapsto \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in \mathcal{F}_k} w_k.$$

The *toric ring* $T_{\mathcal{P}}$ associated to $\mathcal{P}$ is defined as $T_{\mathcal{P}} = \mathbb{K}[\alpha(a) | a \in V(\mathcal{P})] \subset \mathbb{K}[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K]$. The homomorphism

$$\varphi : S \longrightarrow T_{\mathcal{P}}$$
$$x_a \longmapsto \alpha(a)$$

is surjective, and the *toric ideal* $J_{\mathcal{P}}$ is the kernel of φ. The toric ring $T_{\mathcal{P}}$ is viewed as a standard graded $\mathbb{K}$-algebra and, therefore, the corresponding toric ideal $J_{\mathcal{P}}$ is standard graded.

By definition, $J_{\mathcal{P}}$ is a prime ideal containing $I_{\mathcal{P}}$. Moreover, the next result shows that for any polyomino $\mathcal{P}$, $(J_{\mathcal{P}})_2$, the homogeneous part of degree 2 of $J_{\mathcal{P}}$, is equal to $I_{\mathcal{P}}$, which means that the minimal generators of $I_{\mathcal{P}}$ are all and only the minimal generators of degree 2 of $J_{\mathcal{P}}$.

LEMMA 3.1
Let $\mathcal{P}$ be a polyomino. Then $I_{\mathcal{P}} = (J_{\mathcal{P}})_2$.

Proof
First of all, we show that $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$. Let $f \in \mathcal{M}$, with $f = x_a x_b - x_c x_d$. Since $[a,b]$ is an inner interval of $\mathcal{P}$, the corners a and d (resp. b and c) lie on the same horizontal edge interval H_i (resp. H_j). In the same way, it holds that a and c (resp. b and d) lie on the same vertical edge interval V_l (resp. V_m). Therefore,

$$\varphi(x_a x_b) = h_i h_j v_l v_m \prod_{k=1,\dots,r} w_k^{p_k} \tag{1}$$

and

$$\varphi(x_c x_d) = h_i h_j v_l v_m \prod_{k=1,\dots,r} w_k^{n_k} \tag{2}$$

for some $p_k, n_k \in \{0,1,2\}$. We have to show that for any $k \in \{1,\dots,r\}$ $p_k = n_k$. If $\mathcal{P}$ has no holes, then $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in J_{\mathcal{P}}$. Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_r$ are holes of $\mathcal{P}$ and consider $\mathcal{H}_k$ for $k = 1,\dots,r$. Observe that the left lower corner e_k of $\mathcal{H}_k$ satisfies one of the following:

(1) $e_k < a$;
(2) $a \le e_k \le d$;
(3) $d < e_k$.

Case (1). w_k does not divide $\varphi(f)$ (see Figure 1(1)).
Case (2). w_k either divides both $\varphi(x_a)$ and $\varphi(x_c)$ (see Figure 1(2)) or it does not divide $\varphi(x_a x_b)$ or $\varphi(x_c x_d)$.
Case (3). w_k either divides $\varphi(x_a)$ and $\varphi(x_d)$ (see Figure 1(3A)) or all $\varphi(x_a), \varphi(x_b), \varphi(x_c)$ and $\varphi(x_d)$ (see Figure 1(3B)), or w_k does not divide $\varphi(x_a x_b)$ or $\varphi(x_c x_d)$.

Therefore, $n_k = p_k$, and it holds for any $k = 1,\dots,r$. It follows $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in \ker\varphi = J_{\mathcal{P}}$. Since all generators of $I_{\mathcal{P}}$ belong to $J_{\mathcal{P}}$, the inclusion $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$ is proved.

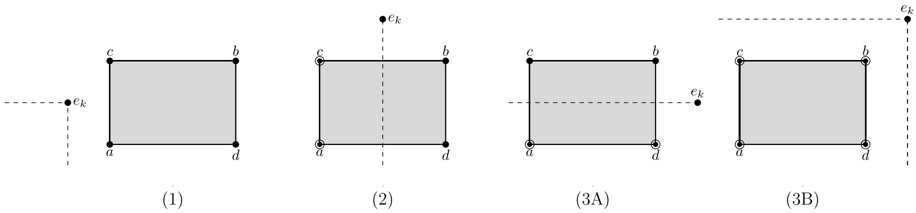


Figure 1. Some positions of e_k and induced flagging on $[a,b]$.

We are going to prove the other inclusion—namely, $(J_{\mathcal{P}})_2 \subseteq I_{\mathcal{P}}$. Let $f \in J_{\mathcal{P}}$, $f = x_a x_b - x_c x_d$. We start by observing that if $a = b$ or $a \in \{c,d\}$, we obtain that f is null. Hence, we assume without loss of generality that $a < b$ and $c < d$. Since $\varphi(x_a x_b) = \varphi(x_c x_d)$, by (1) and (2) the vertices a and d (resp. b and c) lie on the same horizontal edge interval of $\mathcal{P}$; a and c (resp. b and d) lie on the same vertical edge interval of $\mathcal{P}$; and all the vertices of these edge intervals belong to $\mathcal{P}$. Therefore, the vertices a,b,c, and d are the corners of the interval $[a,b]$. By contradiction, we assume that $[a,b]$ is not an inner interval of $\mathcal{P}$; namely, there exists a set of cells $\mathcal{C}$ that does not belong to $\mathcal{P}$ such that $[a,b] \cap \mathcal{C} \neq \emptyset$. We observe that the set $[a,b] \cap \mathcal{C}$ is a set of holes of $\mathcal{P}$ properly contained in $[a,b]$ because $[a,d]$, $[a,c]$, $[b,c]$, and $[b,d]$ are edge intervals in $\mathcal{P}$. Let $\mathcal{H}_1$ be a hole in $[a,b] \cap \mathcal{C}$ with lower left corner $e = (i,j)$. Let $\mathcal{F}_1 = \{(m,n) \in V(\mathcal{P}) | m \leq i \text{ and } n \leq j\}$, then a is the unique vertex in $\{a,b,c,d\}$ such that $a \in \mathcal{F}_1$; namely, $w_1 | \varphi(x_a x_b)$ but $w_1 \nmid \varphi(x_c x_d)$, and $f \notin J_{\mathcal{P}}$. The assertion follows. □

Completely describing the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ is not an easy task. However, if the polyomino contains a particular collection of inner intervals, then we have some partial information on the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$. The latter also gives a sufficient condition for the nonprimality of $I_{\mathcal{P}}$—hence, a necessary condition for the primality. In the rest of the section, we give such a condition.

DEFINITION 3.2

Let $\mathcal{P}$ be a polyomino. A sequence of distinct inner intervals $\mathcal{W} : I_1, \dots, I_\ell$ of $\mathcal{P}$ such that v_i, z_i are diagonal (resp. anti-diagonal) corners and u_i, v_{i+1} the anti-diagonal (resp. diagonal) corners of I_i, for $i = 1, \dots, \ell$, is a *zig-zag walk* of $\mathcal{P}$, if

- (Z1) $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$, for $i = 1, \dots, \ell - 1$;
- (Z2) v_i and v_{i+1} are on a same edge interval of $\mathcal{P}$, for $i = 1, \dots, \ell$;
- (Z3) for any $i, j \in \{1, \dots, \ell\}$, with $i \neq j$, does not exist an inner interval J of $\mathcal{P}$ such that $z_i, z_j \in J$.

REMARK 3.3

Let $\mathcal{W} : I_1 \dots, I_\ell$ be a zig-zag walk of $\mathcal{P}$. Then

1. if v_i is a diagonal vertex of I_i, then v_{i+1} is an anti-diagonal vertex of I_{i+1};
2. ℓ is even.

Proof

(1) Assume that v_k, with $k \in \{1, \dots, \ell - 1\}$, is a diagonal corner of I_k. From condition (Z2), v_{k+1} lies on the same edge interval of v_k, say E, and is an anti-diagonal corner of I_k. The line containing E divides $\mathbb{N}^2$ in two semi-planes. From condition (Z1), we have $I_k \cap I_{k+1} = \{v_{k+1}\}$; hence, I_k and I_{k+1} do not lie on the same semi-plane. Therefore, v_{k+1} is an anti-diagonal corner of I_{k+1}, as well. Observe that the latter justifies the name "zig-zag."

(2) Assume that the starting point v_1 is a diagonal corner of I_1. From (1), it follows that the vertex v_k is a diagonal corner of I_k if and only if k is even (resp. anti-diagonal corner if and only if k is odd). Since $v_{\ell+1} = v_1$, $\ell + 1$ is odd. □

REMARK 3.4

Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}} \subset S$ the polyomino ideal associated to $\mathcal{P}$. If $f \in I_{\mathcal{P}}$, then

$$f = \sum f_{I_i} f_i = \sum x_{a_i} x_{b_i} f_i - \sum x_{c_i} x_{d_i} f_i,$$

where $f_{I_i} = x_{a_i} x_{b_i} - x_{c_i} x_{d_i} \in \mathcal{M}$; hence, for every m, monomial of f, there are two variables in m that are (anti-)diagonal corners of an inner interval of $\mathcal{P}$.

The following proposition gives a necessary condition on $\mathcal{P}$ to be a nonprime polyomino ideal $I_{\mathcal{P}}$.

PROPOSITION 3.5

Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk $\mathcal{W} : I_1, \dots, I_\ell$ in $\mathcal{P}$, then

$$x_{v_1}, \dots, x_{v_\ell} \quad \textit{and} \quad f_{\mathcal{W}} = \prod_{k=1,\dots,\ell} x_{z_k} - \prod_{j=1,\dots,\ell} x_{u_j}$$

are zero divisors of $K[\mathcal{P}]$ with $x_{v_i} f_{\mathcal{W}} \in I_{\mathcal{P}}$ for $i = 1, \dots, \ell$.

Proof

For any vertex v_j in $v_1, \dots, v_\ell$, after relabeling, we may assume $j = 1$. Let $f_{I_i} \in \mathcal{M}$ be associated to the inner interval I_i.

We define the following polynomial:

$$\tilde{f} = \prod_{k>1} x_{z_k} f_{I_1} + \cdots + (-1)^{i+1} \prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} f_{I_i} + \cdots + (-1)^{\ell+1} \prod_{j<\ell} x_{u_j} f_{I_\ell}.$$

Let $i = 1, \dots, \ell - 1$. Suppose that v_i is a diagonal corner of I_i; hence, v_{i+1} is an anti-diagonal corner of I_{i+1}. It holds that

$$\begin{aligned} &\prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} f_{I_i} - \prod_{j<i+1} x_{u_j} \prod_{k>i+1} x_{z_k} f_{I_{i+1}} \\ &\quad = \prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} (x_{v_i} x_{z_i} - x_{v_{i+1}} x_{u_i}) \end{aligned}$$

$$- \prod_{j<i+1} x_{u_j} \prod_{k>i+1} x_{z_k}(x_{v_{i+2}} x_{u_{i+1}} - x_{v_{i+1}} x_{z_{i+1}})$$

$$= \prod_{j<i} x_{u_j} \prod_{k\geq i} x_{z_k} x_{v_i} - \prod_{j\leq i+1} x_{u_j} \prod_{k>i+1} x_{v_{i+2}}.$$

Due to the alternation of the signs in $\tilde{f}$ and by Remark 3.3, it follows that

$$\tilde{f} = \pm\Big(\prod_{k=1,\dots,\ell} x_{z_k} x_{v_1} - \prod_{j=1,\dots,\ell} x_{u_j} x_{v_1} \Big) = \pm x_{v_1} f_{\mathcal{W}},$$

and the sign of $\tilde{f}$ depends on whether v_1 is a diagonal corner in I_1.

Since $\tilde{f}$ is sum of polynomials in $I_{\mathcal{P}}$, then $\tilde{f} \in I_{\mathcal{P}}$. Observe that, by hypothesis, for $i \neq j$, z_i, z_j do not belong to the same inner interval of $\mathcal{P}$, and the same fact holds for u_i and u_j, with $i \neq j$. Due to this fact and by Remark 3.4, $f \notin I_{\mathcal{P}}$. Therefore, x_{v_1} and $f_{\mathcal{W}}$ are zero divisors of $K[\mathcal{P}]$. □

COROLLARY 3.6

Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk in $\mathcal{P}$, then $I_{\mathcal{P}}$ is not prime.

REMARK 3.7

The ideal $J_{\mathcal{P}}$ contains the binomials associated to zig-zag walks. Indeed, let $\mathcal{W}$ be a zig-zag walk and let $f_{\mathcal{W}}$ be its associated binomial. From the proof of Proposition 3.5, it arises that

$$x_{v_1} f_{\mathcal{W}} \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$$

and, due to primality of $J_{\mathcal{P}}$, it follows that $f_{\mathcal{W}} \in J_{\mathcal{P}}$.

We give an example to better understand the structure of $J_{\mathcal{P}}$.

EXAMPLE 3.8

We consider the polyomino in Figure 2. By using `Macaulay2`, we computed the ideal $J_{\mathcal{P}}$ associated to $\mathcal{P}$. $J_{\mathcal{P}}$ has 50 generators, 46 having degree 2, corresponding to the inner 2-minors of $\mathcal{P}$; and 4 having degree 4 that do not belong to $I_{\mathcal{P}}$. The latter are

$$f_1 = x_{(1,3)} x_{(3,1)} x_{(7,4)} x_{(8,2)} - x_{(1,2)} x_{(3,4)} x_{(7,1)} x_{(8,3)},$$

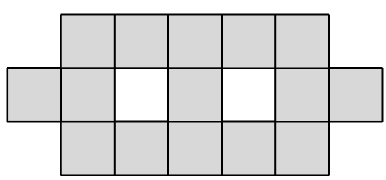


Figure 2. A nonprime polyomino.

Figure 3. The zig-zag walks related to $f_1, \dots, f_4$.

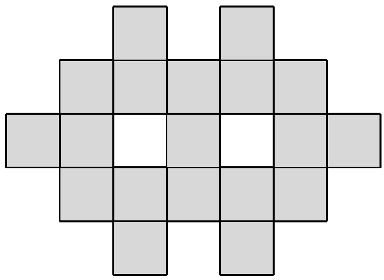


Figure 4. A nonprime polyomino $\mathcal{P}$ such that not all the generators of $J_{\mathcal{P}}$ are related to zig-zag walks.

$$f_2 = x_{(1,3)}x_{(2,1)}x_{(7,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(7,1)}x_{(8,3)},$$
$$f_3 = x_{(1,3)}x_{(3,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(3,4)}x_{(6,1)}x_{(8,3)},$$
$$f_4 = x_{(13)}x_{(2,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(6,1)}x_{(8,3)}.$$

The four binomials above correspond to the four zig-zag walks drawn in Figure 3.

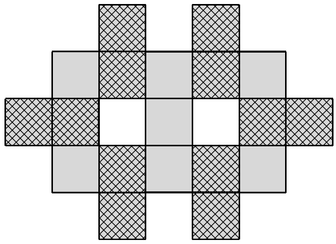
In this case, the generators of $J_{\mathcal{P}}$ in $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ are all related to zig-zag walks. However, we computed $J_{\mathcal{P}}$ for the polyomino in Figure 4, and we found that there are generators of degree 6 that are not related to zig-zag walks; for example,

$$g = x_{(1,4)}x_{(3,1)}x_{(4,6)}x_{(5,1)}x_{(6,6)}x_{(8,3)} - x_{(1,3)}x_{(3,6)}x_{(4,1)}x_{(5,6)}x_{(6,1)}x_{(8,4)}.$$

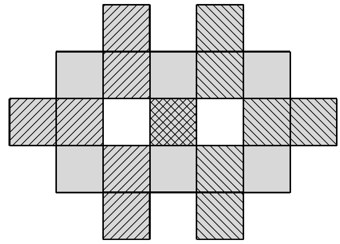
In Figure 5(A), we highlight the intervals related to g. Moreover, there are two zig-zag walks that arises from g, as in Figure 5(B).

It is not an easy task to verify that the nonexistence of a zig-zag walk is a sufficient condition for the primality of $I_{\mathcal{P}}$ for any multiply connected polyomino $\mathcal{P}$ of rank $\leq$ 14. In fact, the set of polyominoes grows exponentially with respect to the rank as the following table, obtained by the implementation in [6], shows.

Rank	7	8	9	10	11	12	13	14
Multiply connected polyominoes	1	6	37	195	979	4663	21474	96496



(A) g is not related to a zig-zag walk...



(B) ...but there are two zig-zag walks

Figure 5

THEOREM 3.9
Let $\mathcal{P}$ be a polyomino with $\operatorname{rank}(\mathcal{P}) \leq 14$. *The following conditions are equivalent:*

(1) *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
(2) *$\mathcal{P}$ contains no zig-zag walks.*

Proof
(1) $\Rightarrow$ (2) It is an immediate consequence of Corollary 3.6.

(2) $\Rightarrow$ (1) To prove the claim we have implemented a computer program that performs the following three steps:

(S1) Compute the set of all multiply connected polyominoes with rank ≤ 14; namely, P.
(S2) Compute the set of polyominoes $\mathrm{NP} \subset P$ whose associated ideals are not primes. We used a routine developed in `Macaulay2` (see [2]).
(S3) Verify that all polyominoes in NP have at least one zig-zag walk.

We refer readers to [6] for a complete description of the algorithm that we used. □

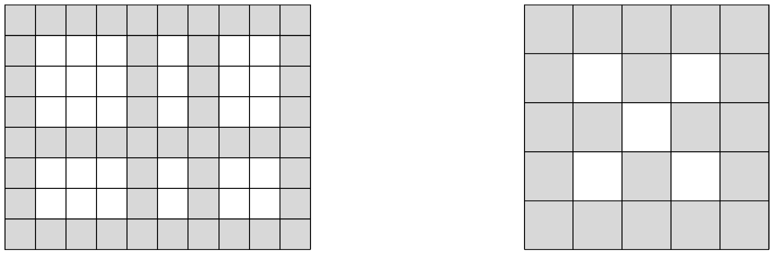
4. Grid polyominoes

From a view point of finding a new class of prime polyomino ideals, due to Corollary 3.6, it is reasonable to consider multiply connected polyominoes with no zig-zag walks. In this section, we consider polyominoes obtained from subtracting some inner intervals by a given interval of $\mathbb{N}^2$, similar to what was done in [5] and [9]. But, if the cells are removed without a specific pattern, one can easily obtain a zig-zag walk in this case, too (see Figure 6(B)). Hence, we define an infinite family of polyominoes with no zig-zag walks by their intrinsic shape: the grid polyominoes.

DEFINITION 4.1
Let $\mathcal{P} \subseteq I := [(1,1),(m,n)]$ be a polyomino such that

$$\mathcal{P} = I \setminus \{\mathcal{H}_{ij} : i \in [r], j \in [s]\},$$

where $\mathcal{H}_{ij} = [a_{ij}, b_{ij}]$, with $a_{ij} = ((a_{ij})_1, (a_{ij})_2)$, $b_{ij} = ((b_{ij})_1, (b_{ij})_2)$, $1 < (a_{ij})_1 < (b_{ij})_1 < m$, $1 < (a_{ij})_2 < (b_{ij})_2 < n$, and



(A) A grid polyomino

(B) A non-grid polyomino, with a zig-zag walk.

Figure 6. A grid polyomino and a non-grid polyomino.

(1) for any $i \in [r]$ and $\ell, k \in [s]$ we have $(a_{i\ell})_1 = (a_{ik})_1$ and $(b_{i\ell})_1 = (b_{ik})_1$;
(2) for any $j \in [s]$ and $\ell, k \in [r]$ we have $(a_{\ell j})_2 = (a_{kj})_2$ and $(b_{\ell j})_2 = (b_{kj})_2$;
(3) for any $i \in [r-1]$ and $j \in [s-1]$, we have $(a_{i+1j})_1 = (b_{ij})_1 + 1$ and $(a_{ij+1})_2 = (b_{ij})_2 + 1$.

We call $\mathcal{P}$ a *grid polyomino*.

Let $\mathcal{P}$ be a grid polyomino and let $T_{\mathcal{P}}$ and $J_{\mathcal{P}}$ be the toric ring and the toric ideal associated to $\mathcal{P}$, respectively, as defined in Section 3, where the hole H_{ij} induces the subset $\mathcal{F}_{i,j}$ and the variable $\omega_{i,j}$. We claim that the grid polyominoes are primes. In order to prove this, we are going to show that $I_{\mathcal{P}} = J_{\mathcal{P}}$.

Let $f = f^+ - f^- \in J_{\mathcal{P}}$. We define that $V_+ = \{v \in V(\mathcal{P}) \mid x_v \text{ divides } f^+\}$, and, similarly, that $V_- = \{v \in V(\mathcal{P}) \mid x_v \text{ divides } f^-\}$. A binomial f in a binomial ideal J is said to be *redundant* if it can be expressed as a linear combination of binomials in J of lower degree. A binomial is said to be *irredundant* if it is not redundant. The following lemma, which was stated in [9] but only for a family of polyominoes, holds also for any $J_{\mathcal{P}}$, as defined in Section 3. Even if the proof is essentially the same as in [9, Lemma 2.2], we report it for the sake of completeness.

LEMMA 4.2
Let $f = f^+ - f^- \in J_{\mathcal{P}}$ be a binomial of degree ≥ 3. If there exist three vertices $p, q \in V_+$ and $r \in V_-$ such that p, q are diagonal (resp. anti-diagonal) corners of an inner interval of $\mathcal{P}$ and r is one of the anti-diagonal (resp. diagonal) corners of the inner interval, then f is redundant in $J_{\mathcal{P}}$.

Proof
Let s be the other corner of the inner interval determined by p, q and r. Then

$$\begin{aligned} f &= f^+ - f^- \\ &= x_p x_q \frac{f^+}{x_p x_q} - f^- \end{aligned}$$

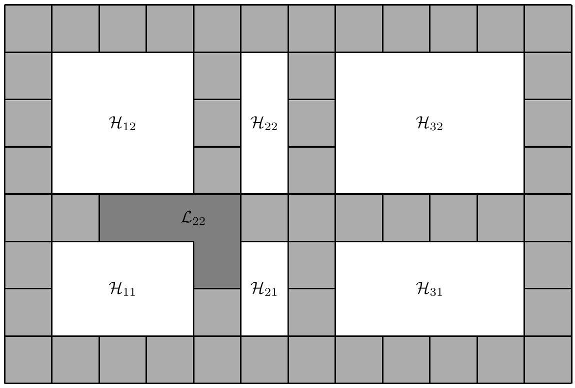


Figure 7. An example of $\mathcal{L}_{i,j}$.

$$= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r x_s \frac{f^+}{x_p x_q} - f^-$$
$$= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r \Big(x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r}\Big).$$

By Lemma 3.1, it holds that $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. Since $x_p x_q - x_r x_s \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$, and $J_{\mathcal{P}}$ is a prime ideal, then $x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \in J_{\mathcal{P}}$, and the statement is proved. □

Let $\mathcal{P}$ be a grid polyomino, and let $\mathcal{H}_{ij}$, for $i \in [r]$ and $j \in [s]$, be its holes, enumerated as in Definition 4.1. Fix $i \in [r]$ and $j \in [s]$. We denote by $\mathcal{L}_{i,j}$ the set

$$\mathcal{L}_{i,j} = \mathcal{F}_{i,j} \setminus \bigcup_{\substack{k \le i \\ h \le j \\ (h,k) \ne (i,j)}} \mathcal{F}_{h,k}.$$

Figure 7 displays an example of a set $\mathcal{L}_{i,j}$. In particular, for the grid polyomino $\mathcal{P}$ in the figure, $\mathcal{L}_{2,2}$ consists of all vertices of $\mathcal{P}$ in the dark grey region.

LEMMA 4.3

Let $\mathcal{P}$ be a grid polyomino. Let $f = f^+ - f^- \in J_{\mathcal{P}}$. If $v \in V_+ \cap \mathcal{L}_{i,j}$, for some $i \in [r]$ and $j \in [s]$, then there exists $v' \in V_- \cap \mathcal{L}_{i,j}$.

Proof

We prove the assertion showing that for all (i, j) and any $v \in \mathcal{L}_{i,j}$ with $v \in V_+$, there exists $v' \in V_-$ such that $v' \in \mathcal{L}_{i,j}$. Let

$$(i_1, j_1) = \min\{(k, h) \mid V_+ \cap \mathcal{F}_{k,h} \ne \emptyset\}.$$

If such a pair does not exist, there is nothing to prove. Otherwise, let $v_1 \in V_+ \cap \mathcal{L}_{i_1,j_1}$. Since $\omega_{i_1,j_1} \mid \varphi(f^+)$, then $\omega_{i_1,j_1} \mid \varphi(f^-)$. It follows that there exists $v_1' \in V_- \cap \mathcal{F}_{i_1,j_1}$.

By the minimality of the pair (i_1, j_1) and since $\varphi(f^+) = \varphi(f^-)$, $v_1' \in \mathcal{L}_{i_1,j_1}$. Let

$$(i_2, j_2) = \min\big\{(k,h) \mid \big(V_+ \setminus \{v_1\}\big) \cap \mathcal{F}_{k,h} \neq \emptyset\big\}.$$

If such a pair does not exist, we are done. Otherwise, let $v_2 \in (V_+ \setminus \{v_1\}) \cap \mathcal{L}_{i_2,j_2}$. We observe that because of the existence of v_1 and v_1', we have the following equation:

$$f = \Big(\prod_{\substack{k \geq i_1 \\ h \geq j_1}} \omega_{k,h}\Big) g,$$

where we have collected all $\omega_{k,h}$'s induced by v_1 and v_1'. Because of the existence of v_2, we have that

$$\omega_{i_2,j_2} \mid \varphi(g^+) = \varphi(g^-).$$

It follows that there exists $v_2' \in (V_- \setminus \{v_1'\}) \cap \mathcal{F}_{i_2,j_2}$. By the minimality of the pair (i_2, j_2), $v_2' \in \mathcal{L}_{i_2,j_2}$. Iterating this procedure, the assertion follows. □

THEOREM 4.4
Let $\mathcal{P}$ be a grid polyomino. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.

Proof
By Lemma 3.1, $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We have to prove the opposite inclusion; that is, $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. Since $(J_{\mathcal{P}})_2 = I_{\mathcal{P}}$, it suffices to prove that any irredundant binomial of $J_{\mathcal{P}}$ is of degree 2. Let $f = f^+ - f^- \in J_{\mathcal{P}}$, with $\deg(f) \geq 3$. Assume by contradiction that f is irredundant. First, we show that there is no $v \in (V_+ \cup V_-) \cap \mathcal{F}$, where $\mathcal{F} = \bigcup_{i \in [r], j \in [s]} \mathcal{F}_{i,j}$. Assume by contradiction that there exists $v_1 \in (V_+ \cup V_-) \cap \mathcal{F}$. In particular, $v_1 \in \mathcal{L}_{i_1,j_1}$, for some $i_1 \in [r]$, $j_1 \in [s]$. Without loss of generality, we may assume $v_1 \in V_+$. By Lemma 4.3, there exists $v_1' \in V_- \cap \mathcal{L}_{i_1,j_1}$. Note that, by condition (3) in Definition 4.1, v_1 belongs to $V(\mathcal{P}) \cap V(\mathcal{H}_{ij})$, for some i, j. The same holds for v_1'. Assume $v_1 < v_1'$. We have the following three cases:

(1) v_1 and v_1' belong to the same maximal vertical (resp. horizontal) edge interval;
(2A) at least one of the corners v_1 and v_1' is not a corner of a hole of $\mathcal{P}$ (e.g., see Figure 8(A));
(2B) v_1 and v_1' are both diagonal (or anti-diagonal) corners of some holes of $\mathcal{P}$ (e.g., see Figure 8(B)).

(1) If v_1 and v_1' belong to the same maximal vertical edge interval, there exists $v_2' \in V_-$ that lies on the same maximal horizontal edge interval of v_1. The vertices v_1, v_1', and v_2' are corners of an inner interval of $\mathcal{P}$, and by Lemma 4.2, f is redundant, which is a contradiction. Similarly, we see that v_1 and v_1' do not belong to the same maximal horizontal edge interval.

(2A) We assume that at least one of the corners v_1 and v_1' is not a corner of a hole of $\mathcal{P}$; we say v_1. Denote by v_2' and v_3' the vertices in V_- that belong to the same horizontal and vertical edge interval of v_1, respectively. The vertices v_1, v_2', v_3' are corners of an

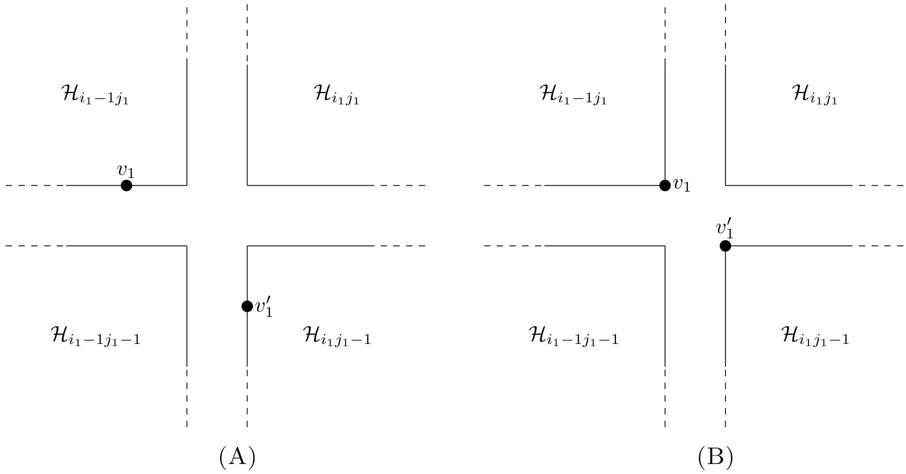


Figure 8. Some possible positions of v_1 and v'_1.

inner interval of $\mathcal{P}$; hence, by applying Lemma 4.2 to v_1, v'_2, v'_3, we obtain that f is redundant, which is a contradiction.

(2B) We denote by v'_2 the vertex in V_- that belongs to the same vertical edge interval of v_1. The vertices v'_1 and v'_2 are diagonal (or anti-diagonal) corners of an inner interval of $\mathcal{P}$. Denote by g, h the other two corners, where g is the one on the same horizontal edge interval of v'_1. Then the binomial $x_{v'_1}x_{v'_2} - x_g x_h \in J_{\mathcal{P}}$, and

$$\begin{aligned}
f &= f^+ - f^- \\
&= f^+ - x_{v'_1}x_{v'_2}\frac{f^-}{x_{v'_1}x_{v'_2}} \\
&= f^+ - x_h x_g\Big(\frac{f^-}{x_{v'_1}x_{v'_2}}\Big) - (x_{v'_1}x_{v'_2} - x_g x_h)\frac{f^-}{x_{v'_1}x_{v'_2}} \\
&= f' - (x_{v'_1}x_{v'_2} - x_g x_h)\frac{f^-}{x_{v'_1}x_{v'_2}}.
\end{aligned}$$

Let v'_3 be the vertex in V_- that belongs to the same horizontal edge interval of v_1. The vertices v_1, v'_3, and g are corners of an inner interval of $\mathcal{P}$. Since $f' \in J_{\mathcal{P}}$, by applying Lemma 4.2 to v_1, v'_3 and g, we obtain that f' is redundant, and then f is also redundant, which is a contradiction.

It follows that the vertices appearing in $V_+ \cup V_-$ do not belong to $\mathcal{F}$. This means $f \in J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ which consists of all cells of $\mathcal{P}$ having no vertices belonging to $\mathcal{F}$. $\mathcal{P}'$ is a simple polyomino and $I_{\mathcal{P}'} = I_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Note that $\alpha(v)$, for every $v \in V(\mathcal{P}) \setminus \mathcal{F}$, is a monomial of degree 2 determined by the maximal horizontal and vertical edge intervals to which v belongs. Then, by [8, Theorem 2.2], $I_{\mathcal{P}'} = J_{\mathcal{P}'} = J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Hence, if f is irredundant in $J_{\mathcal{P}}$, then it is also irredundant in $J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$.

But $I_{\mathcal{P}'}$ is generated by binomials of degree 2; therefore, f is redundant in $I_{\mathcal{P}'}$, and then in $J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$, which is a contradiction. $\square$

COROLLARY 4.5
Let $\mathcal{P}$ be a grid polyomino. Then $I_{\mathcal{P}}$ is prime.

From the main results of this work, which are Corollary 3.6, Theorem 3.9, and Corollary 4.5, the following arises naturally:

CONJECTURE 4.6
Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:

(1) *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
(2) *$\mathcal{P}$ contains no zig-zag walks.*

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