Synthesis of Oldham Mechanisms for Three-Finitely Separated Positions

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Abstract. This paper deals with the synthesis of Oldham mechanisms with inclined and/or orthogonal slots for three-finitely separated positions. In particular, the pole triangle and its circumcircle, along with the corresponding mirror circles, are determined in order to apply the synthesis procedure. The epitrochoid coupler curves are also determined and implemented in a whole synthesis algorithm, which has been validated by means of several significant examples. Moreover, a suitable comparison with the corresponding infinitesimal motion is carried out.

Keywords: Kinematic synthesis · rigid body guidance · double-slider kinematic chain · Oldham mechanisms · coupler curves

1 Introduction

The kinematic synthesis of a rigid body guidance mechanism is related to designing linkages to guide rigid bodies through prescribed positions, with precision and accuracy. This process integrates principles of kinematics and mechanical design to create linkages that ensure controlled movement and the goal is to develop robust mechanisms that fulfill desired motion requirements for various engineering applications [1–6]. These techniques can be applied to any kind of kinematic chains, such as the one of the double-slider mechanisms. These mechanisms are often used in applications where precise linear motion is required, such as in assembly machines, precision instruments, or packaging equipment, but also for the torque transmission between shafts, such as the generalized Oldham coupling [7, 8]. The synthesis of three-poses rigid body guidance mechanisms was first proposed in [9], while this paper is focused on the synthesis of Oldham mechanisms by devoting additional attention to the epitrochoid coupler curves. The rigid body guidance synthesis can be combined with the use of geometric loci, such as the inflection circle, that can be also used as an analysis tool [10, 11]. Another important analysis tool, the centrodes, can offer valuable insights to designers during the synthesis stage providing essential information for mechanism creation [12–15].

This paper deals with the synthesis of Oldham mechanisms with inclined and/or orthogonal slots for three-finitely separated positions. The epitrochoid coupler curves are also determined and implemented in a whole synthesis algorithm, which has been validated by means of several significant examples.

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2 Synthesis of Oldham Mechanisms

Referring to Fig. 1, supposed assigned the three finitely separated positions A_1B_1, A_2B_2 and A_3B_3, the pole triangle P_{12}, P_{23} and P_{13}, its circumcircle, along with the three mirror circles Γ_1, Γ_2 and Γ_3 passing through the cardinal point H_C, which is also the orthocenter of the pole triangle, have been determined for synthesis purposes.

As first reported in [9], the kinematic synthesis of linkages for three-poses rigid-body can be formulated by applying the Suh & Radcliffe method, which makes use of displacement matrices. In fact, a rigid body displacement from a first i-pose to the second j-pose that can be sketched through the segments $A_i - B_i$ and $A_j - B_j$, respectively, is carried out through the homogeneous displacement matrix:

$$\mathbf{D}_{ij} = \begin{bmatrix} \cos\vartheta_{ij} & -\sin\vartheta_{ij} & x_j - x_i\cos\vartheta_{ij} + y_i\sin\vartheta_{ij} \\ \sin\vartheta_{ij} & \cos\vartheta_{ij} & y_j - x_i\sin\vartheta_{ij} - y_i\cos\vartheta_{ij} \\ 0 & 0 & 1 \end{bmatrix} \quad (1)$$

which is obtained by two basic displacement matrices, a rotation matrix of the counter-clockwise angle ϑ_{ij} about the origin O of the fixed frame OXY, and a translation matrix from point A_{iR} that is the image of $A_i = (x_i, y_i)$ after the rotation, up to the point $A_j(x_j, y_j)$. Thus, assigning the position vectors $\boldsymbol{r}_{Ai}$ and $\boldsymbol{r}_{Aj}$ of point A, as follows

$$\mathbf{r}_{Ai} = \left[x_{Ai}\ y_{Ai}\ 1\right]^T \quad \text{and} \quad \mathbf{r}_{Aj} = \left[x_{Aj}\ y_{Aj}\ 1\right]^T \quad (2)$$

along with the relative rotation angle ϑ_{ij}, the displacement matrix $\mathbf{D}_{ij}$ can be obtained by the Eq. (1). Symbol T of Eq. (2) indicates the transpose matrix.

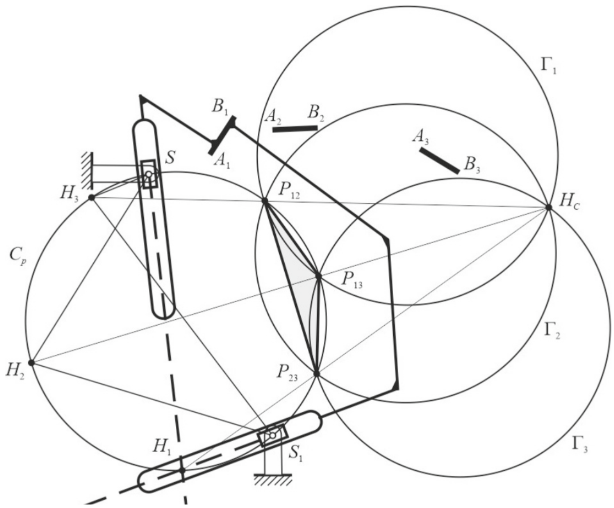

Fig. 1. Graphical synthesis procedure of an Oldham mechanism with inclined slots.

Consequently, the position vector $\mathbf{r}_{Ej}$ of any point E of the rigid body at the j-pose can be determined through the matrix equation:

$$\mathbf{r}_{Ej} = \mathbf{D}_{ij}\ \mathbf{r}_{Ei} \tag{3}$$

Where $\mathbf{r}_{Ei}$ is the position vector of E at the starting position E_i. Of course, the same can be done for the point B, when $\mathbf{r}_{Bi}$ is assigned.

Moreover, the same finite displacement of the rigid body from the i- to the j-pose can be also performed through a pure rotation of the angle ϑ_{ij} about the rotation pole P_{ij}, which position vector $\mathbf{r}_{ij}$ can be expressed in homogeneous coordinates as

$$\mathbf{r}_{ij} = \left[x_{ij}\ y_{ij}\ 1 \right]^T \tag{4}$$

where the Cartesian components x_{ij} and y_{ij} are given by

$$x_{ij} = \frac{x_i + x_j}{2} + \left(y_i - y_j\right)\frac{\sin \vartheta_{ij}}{2\left(1 - \cos \vartheta_{ij}\right)},\ y_{ij} = \frac{y_i + y_j}{2} + \left(x_j - x_i\right)\frac{\sin \vartheta_{ij}}{2\left(1 - \cos \vartheta_{ij}\right)} \tag{5}$$

Thus, a suitable formulation to determine the pole triangle that is made by the rotation poles P_{12}, P_{23} and P_{13}, its circumcircle, along with the three mirror images of it with respect to each edge of the pole triangle, which also intersect each other at the cardinal point H_C (orthocenter), is developed. In particular, each side of the pole triangle and the position of H_C are obtained by applying the analytical geometry.

The circumcircle Cp of the pole triangle can be formulated by imposing at the circle to pass through the rotation poles $P_{12} = (x_{12}, y_{12})$, $P_{23} = (x_{23}, y_{23})$ and $P_{13} = (x_{13}, y_{13})$, which gives the following linear system of three equations in three unknowns:

$$\mathbf{C}\,\mathrm{X} = \mathbf{G} \tag{6}$$

where $\mathbf{C}$ and $\mathbf{G}$ are given by

$$\mathbf{C} = \begin{bmatrix} x_{12} & y_{12} & 1 \\ x_{23} & y_{23} & 1 \\ x_{13} & y_{13} & 1 \end{bmatrix} \qquad \mathbf{G} = \begin{bmatrix} -\left(x_{12}^2 + y_{12}^2\right) \\ -\left(x_{23}^2 + y_{23}^2\right) \\ -\left(x_{13}^2 + y_{13}^2\right) \end{bmatrix} \tag{7}$$

while the column vector X takes the form

$$\mathrm{X} = \left[a\ b\ c \right]^T \tag{8}$$

where a, b and c are the coefficient of the following algebraic equation of Cp:

$$y^2 + x^2 + ax + by + c = 0 \tag{9}$$

Thus, the linear algebraic system of Eq. (6) can be solved by applying the Cramer method, as follows:

$$a = \frac{1}{\det \mathbf{C}} \det\begin{bmatrix} -\left(x_{12}^2 + y_{12}^2\right) & y_{12} & 1 \\ -\left(x_{23}^2 + y_{23}^2\right) & y_{23} & 1 \\ -\left(x_{13}^2 + y_{13}^2\right) & y_{13} & 1 \end{bmatrix} \quad b = \frac{1}{\det \mathbf{C}} \det\begin{bmatrix} x_{12} & -\left(x_{12}^2 + y_{12}^2\right) & 1 \\ x_{23} & -\left(x_{23}^2 + y_{23}^2\right) & 1 \\ x_{13} & -\left(x_{13}^2 + y_{13}^2\right) & 1 \end{bmatrix} \tag{10}$$

$$c = \frac{1}{\det\ \mathbf{C}} \begin{bmatrix} x_{12} & y_{12} & -(x_{12}^2 + y_{12}^2) \\ x_{23} & y_{23} & -(x_{23}^2 + y_{23}^2) \\ x_{13} & y_{13} & -(x_{13}^2 + y_{13}^2) \end{bmatrix} \tag{11}$$

Similarly, still referring to Fig. 1, the equations of the three mirror circles Γ_1, Γ_2 and Γ_3, which pass through the orthocenter H_C of the pole triangle, are determined in the same way by applying the analytical geometry.

Thus, the kinematic synthesis of Oldham mechanisms is carried out by choosing the two arbitrary points S and S_1 on the circumcircle Cp, which represent the corresponding fixed revolute joints of the two pistons, in order to guide with a one d.o.f., the coupler rigid body by means of the slots of the two prismatic pairs.

Referring to Figs. 1 and 2, the three successive poses I, II and III of the coupler rigid body, which correspond to the three assigned finitely separated positions A_1–B_1, A_2–B_2 and A_3–B_3, are achieved by tracing the three lines passing through H_C and that are normal to each side of the pole triangle, by giving the points H_1, H_2 and H_3, respectively, as intersections of these straight lines with the circumcircle Cp.

Consequently, referring to Fig. 2a, the axes of the two straight slots pass through the points S and H_1, and S_1 and H_1, for the starting pose I and likewise, through S–H_2 and S_1–H_2 for the pose II, and finally, through S–H_3 and S_1–H_3 for the pose III.

Of course, even this graphical synthesis procedure has been formulated in the form of algorithm by applying the analytical geometry with the aim to obtain a whole synthesis algorithm of Oldham mechanisms for three-finitely separated positions.

Moreover, the coupler curves of both points S and S_1 that are considered as belonging to the coupler rigid body with two slots, are determined in vector and matrix form. In particular, taking into account that this rigid body motion can be considered as the inverse motion of the Cardan mechanism, the centrodes are represented by the same Cardan circles, where the internal circle is fixed and having a diameter that is equal to the radius of the external moving circle, by giving a typical epitrochoid motion.

Referring to Fig. 2b, the position vector $\mathbf{c}$ of the center C of Cp is expressed as

$$\mathbf{c}\ = \left[-a/2\ -b/2\ 1\right]^T \tag{12}$$

while the position vector $\mathbf{h}_1^*$ of point H_1^* can be obtained by the vector-loop equation:

$$\mathbf{h}_1^* = \mathbf{c} + \mathbf{r} \tag{13}$$

where the position vector $\mathbf{r}$ is given by

$$\mathbf{r} = \left[r\cos(2\theta + \alpha)\ r\sin(2\theta + \alpha)\ 1\right]^T \tag{14}$$

Thus, substituting Eqs. (12) and (14) into Eq. (13), one has

$$\mathbf{h}_1^* = \left[-a/2 + r\cos(2\theta + \alpha) \;\; -b/2 + r\sin(2\theta + \alpha) \;\; 1 \right]^T \tag{15}$$

where θ is the counterclockwise angle shown in Fig. 2.

The position vectors $\mathbf{s}*$ and $\mathbf{s}_1^*$ of points S^* and S_1^* are obtained by the following loop-closure equations:

$$\mathbf{s}* = \mathbf{c} + \mathbf{r} + \mathbf{h}_S \tag{16}$$

$$\mathbf{s}_1^* = \mathbf{c} + \mathbf{r} + \mathbf{h}_{S1} \tag{17}$$

where the position vectors $\mathbf{h}_S$ and $\mathbf{h}_{S1}$ of points S^* and $S_1{}^*$ with respect to $H_1{}^*$ are given by

$$\mathbf{h}_S = \left[-h_S\cos(\theta - \varphi_1) \;\; -h_S\sin(\theta - \varphi_1) \;\; 1 \right]^T \tag{18}$$

$$\mathbf{h}_{S1} = \left[-h_{S1}\cos(\theta - \varphi_2) \;\; -h_{S1}\sin(\theta - \varphi_2) \;\; 1 \right]^T \tag{19}$$

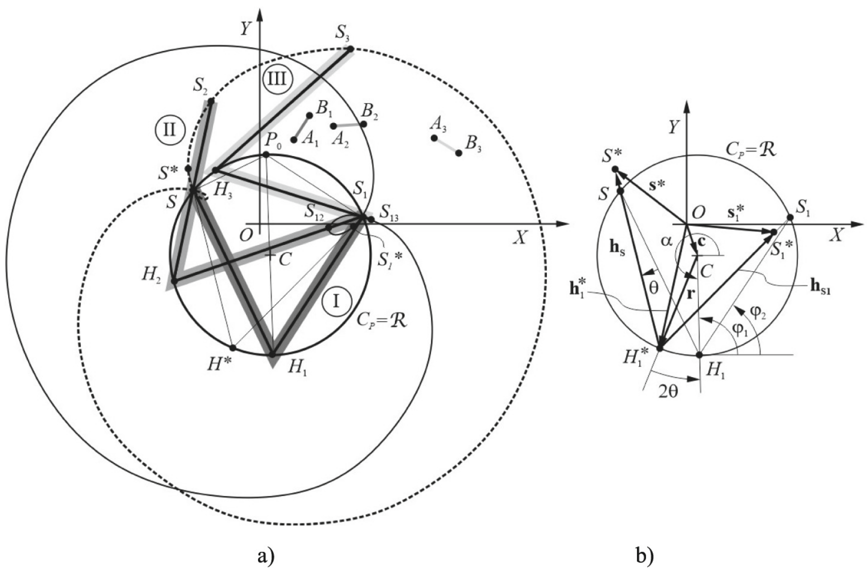

Fig. 2. Scheme of the Oldham mechanism with inclined slots: a) epitrochoid coupler curves of points S and S_1 with cusps b) position vectors and corresponding vector loops.

Therefore, the coupler curves of both points S and S_1 are obtained by substituting the Eqs. (12), (14), (18) and (19) into Eqs. (16) and (17), respectively, as follows

$$\mathbf{s}* = \begin{bmatrix} -a/2 + r\cos(2\theta + \alpha) - h_S\cos(\theta - \varphi_1) \\ -b/2 + r\sin(2\theta + \alpha) - h_S\sin(\theta - \varphi_1) \\ 1 \end{bmatrix} \tag{20}$$

$$\mathbf{s}_1^* = \begin{bmatrix} -a/2 + r\cos(2\theta + \alpha) - h_{s1}\cos(\theta - \varphi_2) \\ -b/2 + r\sin(2\theta + \alpha) - h_{s1}\cos(\theta - \varphi_2) \\ 1 \end{bmatrix} \tag{21}$$

where the angles α, φ_1, φ_2 are shown in Fig. 2b as depending by points H_1, S and S_1.

A suitable comparison with the corresponding infinitesimal motion is now developed by first determining the coordinates of the instant center of rotation P_0 by means of the analytical geometry and thus, as intersecting point of the following straight lines, in agreement with the Chasles theorem:

$$y - y_S = -(1/m_1)(x - x_S) \qquad y - y_{S1} = -(1/m_2)(x - x_{S1}) \tag{22}$$

where the angular coefficients m_1 and m_2, are given by

$$m_1 = (y_S - y_{H1})/(x_S - x_{H1}) \qquad m_2 = (y_{S1} - y_{H1})/(x_{S1} - x_{H1}) \tag{23}$$

Consequently, the position vector $\mathbf{r}_{P0}$ of P_0 takes the form

$$\mathbf{r}_{P0} = \begin{bmatrix} \frac{(m_1 x_{S1} - m_2 x_S) - m_1 m_2 (y_S - y_{S1})}{m_1 - m_2} & \frac{x_S - x_{S1} + m_1 y_S - m_2 y_{S1}}{m_1 - m_2} & 1 \end{bmatrix}^T \tag{24}$$

Thus, the return circle $\mathcal{R}$ for the pose I coincides with Cp and the cross center H_1 of the slots moves along Cp as opposite point of P_0 and thus, representing the return pole.

3 Graphical and Numerical Results

The proposed general algorithm for the synthesis of Oldham mechanisms with inclined and/or orthogonal slots for three-finitely separated positions has been implemented in Matlab, in order to run significant examples for validation purposes. In particular, referring to Fig. 3, the first two examples of Figs. 3a and 3b show the pole triangle and its circumcircle, along with the three mirror circles and the orthocenter, by starting with the knowledge of the three poses A_1B_1, A_2B_2 and A_3B_3, while Figs. 3c and 3d refer to the case of inclined slots and Figs. 3e and 3f refer to the case of orthogonal slots. The corresponding input data are: $A_1 = (30, 120)$ mm, $B_1 = (60, 120)$ mm, $A_2 = (120, 130)$ mm, $B_2 = (135, 104.02)$ mm, $A_3 = (190, 110)$ mm, $B_3 = (184.8, 80.5)$ mm, $\theta_{12} = 60°$; $\theta_{23} = -40°$; $\theta_{13} = -100°$; Case 1: $S = (90, 48.9)$ mm, $S_1 = (120, 38.6)$ mm; Case 2: $S = (100, 49.2)$ mm, $S_1 = (91.7, -15.08)$ mm.

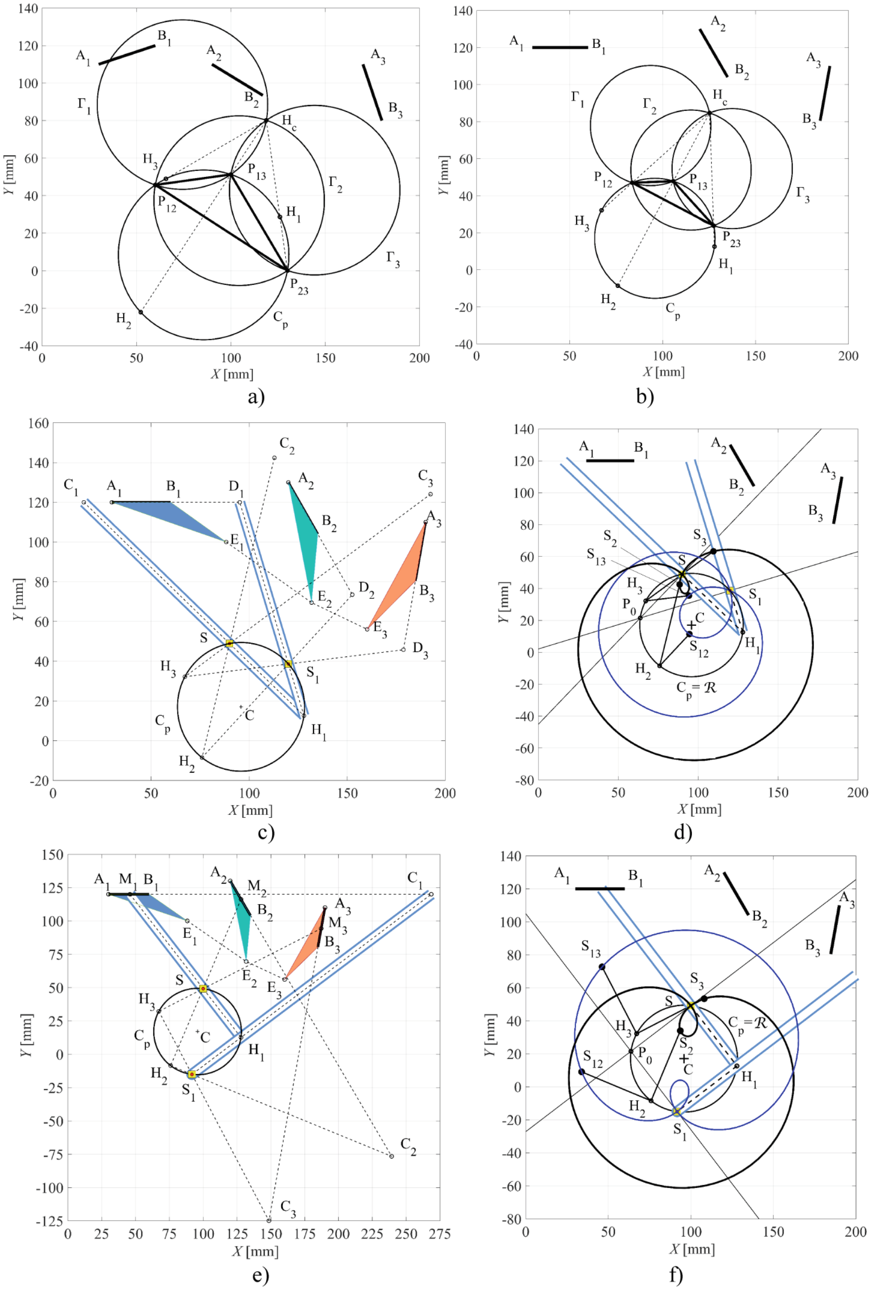

Fig. 3. Synthesis of Oldham mechanisms: a) & b) pole triangle and circles; Case 1: c) inclined slots; d) epitrochoid coupler curves; Case 2: e) orthogonal slots; f) epitrochoid coupler curves.

4 Conclusions

The synthesis of Oldham mechanisms with inclined and/or orthogonal slots for three-finitely separated positions has been developed by formulating a general algorithm, which has been implemented in Matlab, in order to run significant examples and validate the proposed synthesis procedure. In particular, the pole triangle and its circumcircle, along with the corresponding mirror circles, are determined by including the epitrochoid coupler curves of this inverse Cardan motion. Finally, a suitable comparison with the corresponding infinitesimal motion has been carried out for the next developments to design direct and/or inverse Cardan motion mechanisms.

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