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The Logical Sustainability Theory for pension systems: the discrete-time model in a stochastic framework under variable mortality

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Abstract

The aim of this work is to provide the logical sustainability model for defined contribution pension systems (see [1], [2]) in the discrete framework under stochastic financial rate of the pension system fund and stochastic productivity of the active participants. In addition, the model is developed in the assumption of variable mortality tables.

Under these assumptions, the evolution equations of the fundamental state variables, the pension liability and the fund, are provided. In this very general discrete framework, the necessary and sufficient condition of the pension system sustainability, and all the other basic results of the logical sustainability theory, are proved.

In addition, in this work new results on the efficiency of the rule for the stabilization over time of the level of the unfunded pension liability with respect to wages, level that is defined as β indicator, are also proved.

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1 Introduction

Pension systems, designed for public social security purposes, are generally mandatory and substantially pay-as-you-go (PAYG) financed, that is contributions paid by workers are used to finance the pension expenditure for retirees.

In last decades, the demographic trends, especially in Western countries, show a current and future increase in the ratio retirees-workers, which seriously threatens the financial sustainability of PAYG pension systems. Indeed, the sustainability of PAYG pension systems is theoretically based on the results of the Aaron's theorem [3], which is developed under the very strong assumption of a general stability, referred to as a steady state in the economic literature. Currently, in contexts of demographic unbalance and economic instability, that is out of the steady state assumption, the PAYG schemes are unsustainable, and any improvement to their finances can be made by reducing benefits, by increasing contributions (or by a mix of the two), and by postponing the retirement frequently with the same benefit [4, 5].

As a matter of fact, countries with large, unsustainable PAYG systems have enacted reforms of their public pension systems. Substantially, these reforms have postponed the normal retirement age: for example, in the OECD countries the retirement age will increase by 1.5 and 2.1 years on average for men and women, respectively [6]. Other reform options have introduced automatic links between pension benefits and life expectancy, also in defined benefit schemes and in point systems like Germany. On the other hand, more structural reforms, made in recent years, have changed the formula for benefit calculation, moving from the defined benefit to the defined contribution scheme, in particular with the aim of reducing the pension expenditure, and improving transparency in pension accounting [7].

Since the early 1990s, some European countries, among which Sweden and Italy, moved towards the Notional Defined Contribution (NDC) schemes (also referred to as Non-financial Defined Contribution schemes), where contributions are substantially saved in individual accounts and are used to pay current pension benefits according a PAYG logic. To understand conceptual foundations and issues of NDC schemes, and to analyze the pros and cons of these schemes, refer to [8]. In Sweden, the NDC scheme is supplemented by a proper automatic balance mechanism that adjust benefits when the system turns in a situation estimated to be of financial unsustainability. However, despite what many believe, note that the adoption of the NDC scheme does not guarantee the pension system [9].

This paper is inserted in the framework of the logical sustainability theory, which is substantially based on the two forerunner papers [1, 2], where the mathematical formalization of the pension system's structure and the logical management rules that ensure the financial sustainability are set in a "continuous" framework.

The model proposed in the logical sustainability theory considers a defined contribution pension system supported by a funded component. This component, and its related return, are structurally inserted into the management rules of the system. Indeed, in our model, specific conditions of sustainability have been proved, and a proper rule on the rate of return on the pension liability is provided. This rule allows to stabilize the level of the unfunded pension liability in relation to wages, referred to as the beta indicator [10, 11].

Specifically, the aim of this work is twofold: on the one hand, to transfer the logical sustainability model, with its related conditions, from the continuous to the discrete framework; and on the other, to introduce an innovative part, directly developed into the discrete framework, which deals with the efficiency issue of the rule on the rate of return on the pension liability that stabilizes the beta indicator.

In the model transformation from the continuous to the discrete framework, the section concerning the management of a demographic/economic wave will be developed in future research; see [12, 13] for the development of this issue in the continuous framework.

This paper is structured as follows. In Section 2, the definitions of the basic variables and main indicators of the model, and the evolution equations of the fundamental state variables, the pension liability and the fund, are reformulated in the discrete framework. In addition, to take into account the longevity risk that seriously affects the pension system sustainability, in this paper we develop the logical sustainability model under the assumption of variable mortality tables. To take into account updated life tables and to consider the actual group of retirees alive at each time in comparison with the expected group, two new readjustment rates are introduced in the evolution equation of the retiree pension liability. Section 3 contains the transformation from the continuous to the discrete framework of the fundamental conditions for the pension system sustainability, among which the necessary and sufficient condition and the rule for the stabilization of the beta indicator. Hence, the efficiency of this rule is proved in Section 4. Section 5 contains our main concluding comments.

2 The logical sustainability model in the discrete case

In this Section, the logical sustainability model is provided for the discrete case. For a complete description of all the variables and indicators of the logical sustainability model, as well as for a deeper understanding of their meanings, one can refer to [2], where the model is exhaustively illustrated in the continuous framework.

Let $t_* \in \mathbb{R}$ be any initial time, and let one year be the unitary time increment. Hence, the time yearly sequence $\{t_*, t_*+1, t_*+2, t_*+3, ...\}$ is determined. Such sequence can be biunivocally transformed in sequence $\{0, 1, 2, ...k...\}$ by means of transformation $k = t - t_*$.

Throughout the paper we refer to time k, with $k \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of nonnegative integers, or refer to year k, with $k \geq 1$, as the unitary time interval beginning in $k-1$, excluding $k-1$, and ending in k, including k, i.e.

$(k-1, k]$.

2.1 Basic variables

For each time $k \in \mathbb{N}$, the following **state variables** are considered:

1. F_k is the pension system fund, that is the aggregate value of the assets;

2. L_k^A is *the pension liability of contributors*, referred also to as the *latent pension liability*, with $L_k^A \geq 0$;

3. L_k^P is *the pension liability of retirees*, referred also to as the *current pension liability*, with $L_k^P \geq 0$;

4. L_k^T is *the total pension liability*, namely $L_k^T = L_k^A + L_k^P$.

For each time $k \in \mathbb{N}$, with $k \geq 1$, the following **flow (or flow-connected) variables** are considered:

1. α_k is the contribution rate (with $\alpha_k \geq 0$);

2. W_k, C_k, and P_k are the wages, the contributions, and the pension disbursements, with $W_k > 0$, $C_k \geq 0$, and $P_k \geq 0$, respectively; all of them are referred to year k, hence both C_k and P_k are paid in arrears; clearly, it is
$$C_k = \alpha_k W_k; \tag{1}$$

3. ${}^A L_k^P$ is the total yearly pension liability that turns, in time k, from latent into current, after the yearly revaluation by rate r_k^{LA}, see following point 5;

4. r_k is the interest rate returned on fund F_{k-1} for year k; it can be described by a stochastic process [9];

5. r_k^{LA} is the revaluation rate returned on pension liability of contributors L_{k-1}^A for year k;

6. r_k^{LP} is the revaluation rate returned on pension liability of retirees L_{k-1}^P for year k;

7. ${}^* r_k^{LP}$ is the rate explicitly returned on pension liability of retirees L_{k-1}^P for year k.

Refer to following subsection 2.2. for the relationship between r_k^{LP} and ${}^* r_k^{LP}$.

Furthermore, throughout the paper, it is assumed that:

Assumption 1. *For each time k, with $k \geq 1$, the state variables are evaluated after the calculation of the flow variables.*

Specifically, it follows that for each time k, with $k \geq 1$, state variables F_k, L_k^A, L_k^P, and hence L_k^T, are evaluated at the end of year k, immediately after the revenue of annual contribution C_k, the payment of annual pension expenditure P_k, and the transfer of liability from contributors to retirees ${}^{A}L_k^P$.

Assumption 2. *Pension liability L_k^A equals the sum of the contribution amounts of each contributor alive at time k; pension liability L_k^P equals the sum of the liability of each retiree alive at time k. Specifically, the liability of each retiree alive at time k equals the "individual reserve" of a revaluing life annuity, calculated by a zero technical rate and yearly paid in arrears. Therefore, the liability of each retiree results equal to the product of his pension in k times the (curtate) life expectancy calculated with the last life tables available in k.*

2.2 Evolution equations

Firstly, we consider the evolution equation of the fund, whose amount at time $k+1$ is given by

$$F_{k+1} = F_k(1 + r_{k+1}) + C_{k+1} - P_{k+1} \qquad k \in \mathbb{N}. \tag{2}$$

Hence, we consider the evolution equation of the pension liability of contributors, which **in a defined contribution scheme has to be** given by

$$L_{k+1}^A = L_k^A(1 + r_{k+1}^{LA}) + C_{k+1} - {}^{A}L_{k+1}^P \qquad k \in \mathbb{N}. \tag{3}$$

As a consequence of equation (3), it follows that the pension liability, related to contribution amounts of those who have died during their working years, has to be redistributed to other contributors (**the inheritance gain**[1]). In this work, for ease of exposition, and without prejudice to the model proposed, the possible reimbursement of the contribution amount of participants who exit from the pension scheme is not taken into consideration. This case can be easily integrated into our model. It is worth noting that the changing of mortality does not affect the evolution equation of the contributors pension liability, namely equation (3).

Lastly, we consider the **more complex** evolution equation of the pension liability of retirees under the assumption of changing mortality.
In this regard, we denote by ${}_{m}^{h}L_k^P$ the retiree pension liability at time k, evaluated with the last life table available at time h (in our formulation with $h = k$ or $h = k+1$), with reference to the collectivity existing at time m (in our formulation with $m = k$ or $m = k+1$).
By this notation, ${}_{k+1}^{k+1}L_{k+1}^P$ is the pension liability of retirees calculated at time $k+1$, with reference to the collectivity of retirees who are existing to time $k+1$, according to the last life tables available in $k+1$. Analogously, ${}_{k}^{k}L_k^P$ is the pension liability of retirees calculated at time k, with reference to the collectivity of retirees who are existing to time k, according to the last life tables available in k.
Under the assumption of changing life tables and using the above notation, the

[1] In this regard, refer to what is provided in the Swedish pension system, [14] p. 116.

evolution equation of the pension liability of retirees **in a defined contribution scheme is given by**

$$\begin{aligned} {}^{k+1}_{k+1}L^P_{k+1} &= {}^{k}_{k}L^P_{k}(1+H^T_{k+1})(1+H^C_{k+1})(1+{}^*r^{LP}_{k+1}) - P_{k+1} + {}^{A}L^P_{k+1} \\ &\text{for each } k \in \mathbb{N}. \end{aligned} \tag{4}$$

Note that equation (4) shows that the retiree pension liability in $k+1$ stems from both the pension liability of retirees already existing in k, see ${}^{k}_{k}L^P_{k}$ in the first addend at r.h.s of (4), and the pension liability of new retirees, ${}^{A}L^P_{k+1}$, turned into in $k+1$ from the liability of contributors, after the revaluation by rate r^{LA}_{k+1}. In addition, we note that the last life table, available in $k+1$, has also to be used to transform in annuity the pension liability of contributors who have become retirees in $k+1$, whose liability is ${}^{A}L^P_{k+1}$ at r.h.s. of (4).

Referring to equation (4), we have that

a) H^T_{k+1} is the rate of change of retiree pension liability ${}^{k}_{k}L^P_{k}$ as a result of the recalculation of the same with the last life table available in $k+1$. $H^T_{k+1}$is defined by the following equation

$$ {}^{k+1}_{k}L^P_{k} = {}^{k}_{k}L^P_{k}(1+H^T_{k+1}), $$

and hence it results

$$ H^T_{k+1} = \frac{{}^{k+1}_{k}L^P_{k} - {}^{k}_{k}L^P_{k}}{{}^{k}_{k}L^P_{k}}; $$

naturally, if the same life table is used in k and in $k+1$, this rate is zero. We refer to H^T_{k+1} as **the rate of the table readjustment**;

b) H^C_{k+1} is the rate of change of the retiree pension liability evaluated at k, with the last life table available in $k+1$, i.e. ${}^{k+1}_{k}L^P_{k}$, on the basis of the collectivity existing in k and the collectivity effectively existing in $k+1$. H^C_{k+1} is defined by the following equation

$$ {}^{k+1}_{k+1}L^P_{k} = {}^{k+1}_{k}L^P_{k}(1+H^C_{k+1}), $$

and hence it results

$$ H^C_{k+1} = \frac{{}^{k+1}_{k+1}L^P_{k} - {}^{k+1}_{k}L^P_{k}}{{}^{k+1}_{k}L^P_{k}}. $$

Note that ${}^{k+1}_{k}L^P_{k}$ is the reserve for a revaluing life annuity paid in arrears evaluated with respect to the retirees existing in k, whereas ${}^{k+1}_{k+1}L^P_{k}$ equals the reserve, calculated with the last life table available in $k+1$, of a revaluing life annuity paid in advance to retirees effectively existing in $k+1$. In both cases, the reserves have been evaluated before accrediting rate ${}^*r^{LP}_{k+1}$ explicitly returned on the pension liability of retirees; we refer to H^C_{k+1} as **the rate of the collectivity readjustment after the table**

readjustment.
Note that by means of these two rates it is obtained the component of pension liability, ${}^{k+1}_{k+1}L^P_{k+1}$, deriving from the old pension liability related to retirees existing in k, before the rate explicitly returned on the pension liability of retirees in $k+1$, that is

$$ {}^{k+1}_{k+1}L^P_k = {}^{k+1}_{k}L^P_k(1+H^C_{k+1}) = {}^{k}_{k}L^P_k(1+H^T_{k+1})(1+H^C_{k+1}); $$

c) In relation to (4), note that it is similarly possible to define firstly the rate of the collectivity readjustment, denoted by H^{C*}_{k+1}, and secondly the rate of the table readjustment after the collectivity readjustment, denoted by H^{T*}_{k+1}. However, it easy to verify that

$$ (1+H^T_{k+1})(1+H^C_{k+1}) = (1+H^{C*}_{k+1})(1+H^{T*}_{k+1}) \quad \text{for each } k \in \mathbb{N}; $$

d) we set the following relationship between the two rates, r^{LP}_{k+1} and ${}^*r^{LP}_{k+1}$, namely between the yearly revaluation rate of the pension liability of retirees and the yearly rate explicitly returned on the pension liability of retirees,

$$ (1+H^T_{k+1})(1+H^C_{k+1})(1+{}^*r^{LP}_{k+1}) = 1 + r^{LP}_{k+1}. \tag{5} $$

In first approximation, we get the following relationship

$$ r^{LP}_{k+1} \approx H^T_{k+1} + H^C_{k+1} + {}^*r^{LP}_{k+1}, $$

that is the yearly revaluation rate of the pension liability of retirees is approximated by the sum of the above-mentioned three rates.

If we set, by definition, $L^P_k = {}^k_kL^P_k$ for each $k \in \mathbb{N}$, then it is

$$ L^P_{k+1} = L^P_k(1+r^{LP}_{k+1}) - P_{k+1} + {}^A L^P_{k+1} \qquad k \in \mathbb{N}, \tag{6} $$

where r^{LP}_{k+1} is obtained from (5). Referring to the total pension liability, from (3) and (6) it follows

$$ L^T_{k+1} = L^T_k + L^A_k r^{LA}_{k+1} + L^P_k r^{LP}_{k+1} + C_{k+1} - P_{k+1} \qquad k \in \mathbb{N}. $$

If we assume $r^{LA}_k = r^{LP}_k$, $k \in \mathbb{N}$, $k \geq 1$, then we can define the **rate of revaluation of the total pension liability**, denoted by r^L_k, such that

$$ r^L_k = r^{LA}_k = r^{LP}_k \qquad k \in \mathbb{N}. \tag{7} $$

Therefore, it is also

$$ L^T_{k+1} = L^T_k(1+r^L_{k+1}) + C_{k+1} - P_{k+1} \qquad k \in \mathbb{N}. \tag{8} $$

Lastly, assumed W_0 known and $W_0 > 0$, the evolution equation of wages is given by

$$ W_{k+1} = W_k(1+\sigma_{k+1}) \qquad k \in \mathbb{N}, \tag{9} $$

where σ_{k+1} is the growth rate of wages in year $k+1$, subjected to constraint $1+\sigma_{k+1}>0$. If the active population is assumed to be constant in year $k+1$, then it is $\sigma_{k+1}=g_{k+1}$, where g_{k+1} denotes the growth rate of the productivity in year $k+1$. Note that the growth rate of wages can be described by a stochastic process as in [9], for example.
Equation (9) is a linear difference equation, of first order, with variable coefficients, and homogeneous. Set the initial value W_0, there exists a unique solution given by

$$W_k = W_0 \prod_{s=1}^{k} \left(1+\sigma_s\right) \quad k \in \mathbb{N},\ k \geq 1. \tag{10}$$

2.3 Basic definitions

Let us consider time n, with $n \geq 1$.

DEFINITION 1 *A pension system is sustainable in time interval* $[0,n]$ *if and only if* $F_k \geq 0$ *for each* $k=0,1,2,3,...n$.

Note that for each time k pensions are assumed to be paid with reference to the pension liability of retirees at time $k-1$, liability revalued by the rate of return of year k, r_k^{LP}. Hence, the contributions paid at time k are accounted into the pension liability of retirees calculated at time k, and they do not affect the pension payments at time k.

Taking into account this note, referring to the pension payments at time k, for each $k \geq 1$, we refer to the following quantities

$$\begin{gathered} F_{k-1}(1+r_k), \\ L_{k-1}^{A}(1+r_k^{LA}), \\ L_{k-1}^{P}(1+r_k^{LP}), \\ L_{k-1}^{A}(1+r_k^{LA}) + L_{k-1}^{P}(1+r_k^{LP}) \end{gathered}$$

as the **provisional fund**, the **provisional pension liability of contributors**, the **provisional pension liability of retirees**, and the **provisional total pension liability**, respectively.

We consider the following definitions.

DEFINITION 2 *The* ***degree of funding of the pension liability*** *is indicated by* Dc_k *and is given by*

$$Dc_k = \frac{F_k}{L_k^T}.$$

DEFINITION 3 *The* ***provisional degree of funding of the pension liability*** *is indicated by* Dc_k^* *and is given by*

$$Dc_k^* = \frac{F_{k-1}(1+r_k)}{L_{k-1}^T(1+r_k^L)}. \tag{11}$$

Note that the provisional degree of funding is calculated before of the payment of contributions and pensions at time k, whereas the degree of funding is calculated after.
Throughout this paper we assume that $\mathbf{0 \leq F_k \leq L_k^T}$, and we define

DEFINITION 4 *The* ***unfunded and funded pension liability*** *are defined respectively as*

$$L_k^{UN} = L_k^T - F_k \quad and \quad L_k^F = F_k. \tag{12}$$

Under assumption (7), the evolution equation of the unfunded pension liability is obtained from the difference between equations (8) and (2), and it is given by

$$L_{k+1}^{UN} = L_{k+1}^T - F_{k+1} = L_k^{UN} + L_k^T r_{k+1}^L - F_k r_{k+1} \qquad k \in \mathbb{N} \tag{13}$$

or also

$$L_{k+1}^{UN} = L_{k+1}^T - F_{k+1} = L_k^T(1 + r_{k+1}^L) - F_k(1 + r_{k+1}) \qquad k \in \mathbb{N}. \tag{14}$$

Notice that L_{k+1}^{UN} does not depend on the payment of contributions and pensions in $k+1$.
By (12), it follows that the evolution equation of the funded pension liability is provided by the evolution equation of the fund, see (2).

DEFINITION 5 ***(The beta indicator)*** *The* ***level of the unfunded pension liability*** *in relation to wages, the beta indicator, is denoted by β_k and it is*

$$\beta_k = \frac{L_k^{UN}}{W_k}. \tag{15}$$

By definition (12) and using (14), it also follows that

$$\beta_k = \frac{L_k^T - F_k}{W_k} = \frac{L_{k-1}^T(1 + r_k^L) - F_{k-1}(1 + r_k)}{W_k}. \tag{16}$$

It is easy to show that

$$\beta_k = \frac{L_k^T}{W_k}(1 - Dc_k), \tag{17}$$

which makes explicit the link between the two state indicators, β_k and Dc_k.

DEFINITION 6 *The* ***divisor of the provisional total pension liability in the provisional pension liability of retirees*** *is denoted by ν_k and it is*

$$\nu_k = \frac{L_{k-1}^A(1 + r_k^{LA}) + L_{k-1}^P(1 + r_k^{LP})}{L_{k-1}^P(1 + r_k^{LP})}$$

with $L_{k-1}^P \neq 0$.

Note that it is $\nu_k > 1$ generally; however, it is $\nu_k = 1$ if L^A_{k-1} is zeroed, that is when the active group is exhausted. **Under assumption (7)**, it is

$$\nu_k = \frac{L^T_{k-1}}{L^P_{k-1}}.$$

DEFINITION 7 *The **divisor of the provisional pension liability of retirees in the pension expenditure** is denoted by γ_k and it is*

$$\gamma_k = \frac{L^P_{k-1}(1 + r^{LP}_k)}{P_k}$$

with $P_k \neq 0$.

Note that indicator γ_k represents the average duration of the average pension at time k.

DEFINITION 8 *The **divisor of the provisional total pension liability in the pension expenditure** is given by $\gamma_k \nu_k$, i.e.*

$$\gamma_k \nu_k = \frac{L^A_{k-1}(1 + r^{LA}_k) + L^P_{k-1}(1 + r^{LP}_k)}{P_k},$$

with $P_k \neq 0$.

Under assumption (7) it is

$$\gamma_k \nu_k = \frac{L^T_{k-1}(1 + r^L_k)}{P_k},$$

and considering the reciprocals it is

$$\frac{1}{\gamma_k \nu_k} = \frac{P_k}{L^T_{k-1}(1 + r^L_k)}, \tag{18}$$

hence, $\frac{1}{\gamma_k \nu_k}$ can be considered as the yearly rate of repayment of the pension liability (or the yearly pension expenditure paid out per unit of pension liability).

Note that the definition of the $\gamma_k \nu_k$ indicator allows the decomposition of the pension expenditure in its unfunded and covered components. Indeed, we can express the pension expenditure as

$$P_k = \frac{L^T_{k-1}(1 + r^L_k)}{\gamma_k \nu_k} = \frac{L^T_{k-1}(1 + r^L_k) - F_{k-1}(1 + r_k)}{\gamma_k \nu_k} + \frac{F_{k-1}(1 + r_k)}{\gamma_k \nu_k}. \tag{19}$$

Under the assumption that the numerators of both ratios at r.h.s. of (19) are non-negative, we refer to quantities

$$P^{UN}_k = \frac{L^T_{k-1}(1 + r^L_k) - F_{k-1}(1 + r_k)}{\gamma_k \nu_k},$$

$$P_k^C = \frac{F_{k-1}(1+r_k)}{\gamma_k \nu_k}$$

as the **unfunded pension expenditure** and the **covered pension expenditure**, respectively. Hence, by (19) it follows that

$$P_k = P_k^{UN} + P_k^C.$$

DEFINITION 9 *The **degree of PAYG covering of the pension disbursements** is denoted by Dc_k^{PAYG} and is given by*

$$Dc_k^{PAYG} = \frac{C_k}{P_k}. \tag{20}$$

In relation to the contribution rates, that is the pension expenditure with respect to wages, we consider the following definitions.

DEFINITION 10 *The **PAYG contribution rate** is denoted by α_k^{PAYG} and it is*

$$\alpha_k^{PAYG} = \frac{P_k}{W_k}.$$

Hence, α_k^{PAYG} *is the contribution rate that, applied to W_k, makes the corresponding contributions equal to the pension expenditure in k.* Therefore, from (18) it follows that

$$\alpha_k^{PAYG} = \frac{L_{k-1}^T(1+r_k^L)}{\gamma_k \nu_k}\frac{1}{W_k}. \tag{21}$$

DEFINITION 11 *The **level of the unfunded contribution rate**, or the **unfunded contribution rate**, is denoted by α_k^{UN} and it is*

$$\alpha_k^{UN} = \frac{\beta_k}{\gamma_k \nu_k}.$$

From (16), it follows that

$$\alpha_k^{UN} = \frac{L_{k-1}^T(1+r_k^L) - F_{k-1}(1+r_k)}{\gamma_k \nu_k}\frac{1}{W_k}.$$

Hence, α_k^{UN} *is the contribution rate that, applied to W_k, makes the corresponding contributions equal to the unfunded pension expenditure in k.*

DEFINITION 12 *The **level of the covered contribution rate**, or the **covered contribution rate**, is denoted by α_k^C and it is*

$$\alpha_k^C = \frac{F_{k-1}(1+r_k)}{\gamma_k \nu_k}\frac{1}{W_k}.$$

Hence, α_k^C *is the contribution rate that, applied to* W_k*, makes the corresponding contributions equal to the covered pension expenditure in* k*.*
It easy to verify that

$$\alpha_k^{PAYG} = \alpha_k^{UN} + \alpha_k^C.$$

Lastly, with reference to the effective contribution rate α_k, we consider the definition of the **funded contribution rate**:

DEFINITION 13 *The* ***level of the funded contribution rate****, or the* ***funded contribution rate****, is denoted by* α_k^F *and it is*

$$\alpha_k^F = \alpha_k - \alpha_k^{UN} = \alpha_k - \frac{\beta_k}{\gamma_k \nu_k}. \tag{22}$$

We define the **funded contribution**, C_k^F, as

$$C_k^F = (\alpha_k - \alpha_k^{UN})W_k.$$

DEFINITION 14 *The intrinsic rate of return is denoted by* $intr_k$ *and it is defined as the rate such that*

$$1 + intr_k = (1 + r_k)\big(1 - \frac{1}{\gamma_k \nu_k}\big). \tag{23}$$

In first approximation, it is

$$intr_k \approx r_k - \frac{1}{\gamma_k \nu_k}.$$

3 Basic conditions for the pension system sustainability

In the following, **assumption** (7) **is set throughout the paper**.

Let n be any time fixed, with $n \in \mathbb{N}$, $n \geq 1$. We express the necessary and sufficient condition for the pension system sustainability for time interval $[0, n]$, where upper limit n can also be equal to $+\infty$.

THEOREM 1 ***The necessary and sufficient condition for the pension system sustainability.***
Let the pension system have an initial non-negative fund, i.e. $F_0 \geq 0$*.*
The pension system is sustainable in $[0, n]$*, with* $n \geq 1$*, if and only if for each* $k = 1, 2, \ldots n$ *the whole of the annual funded contribution, paid until* k *and discounted at time* 0 *by the intrinsic rate of return, does not create a deficit greater than the initial available fund,* F_0*.*
Hence, assumed $F_0 \geq 0$*, it results*

$$F_k \geq 0 \qquad \textit{for each } k = 1, 2, \dots n$$

if and only if

$$-\sum_{h=1}^{k} W_h\big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=1}^{h} \Big((1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big)\Big)^{-1} \leq F_0 \qquad \textit{for each } k = 1, 2, \dots n. \tag{24}$$

Proof. Let us reformulate the fund evolution equation, see (2), expressing contributions by means of the wages and the contribution rate, namely $C_{k+1} = \alpha_{k+1} W_{k+1}$, and pensions using (18), namely $P_{k+1} = \frac{L_k^T(1+r_{k+1}^L)}{\gamma_{k+1}\nu_{k+1}}$. Hence, we have

$$F_{k+1} = F_k(1+r_{k+1}) + \alpha_{k+1}W_{k+1} - \frac{L_k^T(1+r_{k+1}^L)}{\gamma_{k+1}\nu_{k+1}} \quad k = 0, 1, \dots n-1. \tag{25}$$

Adding and subtracting same quantity $\frac{F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}}$ at the right hand side of (25), we have

$$\begin{aligned} F_{k+1} = {} & F_k(1+r_{k+1}) + \alpha_{k+1}W_{k+1} - \frac{L_k^T(1+r_{k+1}^L)}{\gamma_{k+1}\nu_{k+1}} \\ & + \frac{F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}} - \frac{F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}} \qquad k = 0, 1, \dots n-1, \end{aligned}$$

and hence

$$\begin{aligned} F_{k+1} &= F_k(1+r_{k+1}) + \alpha_{k+1}W_{k+1} - \frac{L_k^T(1+r_{k+1}^L) - F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}} + \\ &- \frac{F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}} = \\ &= F_k(1+r_{k+1}) + \alpha_{k+1}W_{k+1} - \frac{L_{k+1}^{UN}}{\gamma_{k+1}\nu_{k+1}} - \frac{F_k(1+r_{k+1})}{\gamma_{k+1}\nu_{k+1}} \\ &\quad \text{for each } k = 0, 1, \dots n-1. \end{aligned}$$

In this latter, using definition (15) of the beta indicator, after some algebraic calculation, we get the following equation

$$F_{k+1} = F_k(1+r_{k+1})\big(1 - \frac{1}{\gamma_{k+1}\nu_{k+1}}\big) + W_{k+1}(\alpha_{k+1} - \frac{\beta_{k+1}}{\gamma_{k+1}\nu_{k+1}}) \quad \text{for each } k = 0, 1, \dots n-1, \tag{26}$$

where $(1+r_{k+1})\big(1-\frac{1}{\gamma_{k+1}\nu_{k+1}}\big)$ is equal to the compounding factor corresponding to the yearly intrinsic rate defined by (23), and $\alpha_{k+1}-\frac{\beta_{k+1}}{\gamma_{k+1}\nu_{k+1}}$ is the level of

the funded contribution rate defined by (22).
The evolution equation of the fund, see (26), has the following solution

$$F_k = F_0 \prod_{s=1}^{k} (1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big) + \\ + \sum_{h=1}^{k} W_h \big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=h+1}^{k} (1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big) \\ \text{for each } k = 1, 2, \ldots n,$$

and hence

$$F_k = \prod_{s=1}^{k} (1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big) \\ \left(F_0 + \sum_{h=1}^{k} W_h \big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=1}^{h} \Big((1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big)\Big)^{-1}\right) \\ \text{for each } k = 1, 2, \ldots n.$$

It follows that

$$F_k \geq 0 \qquad \text{for each } k = 1, 2, \ldots n$$

if and only if

$$F_0 + \sum_{h=1}^{k} W_h \big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=1}^{h} \Big((1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big)\Big)^{-1} \geq 0 \\ \text{for each } k = 1, 2, \ldots n,$$

and therefore, if and only if (24) holds. □

Remark 1 Taking into account (10), the necessary and sufficient condition can be written as

$$-\sum_{h=1}^{k} \Big(W_0 \prod_{s=1}^{h} (1+\sigma_s)\Big) \big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=1}^{h} \Big((1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big)\Big)^{-1} \leq F_0 \\ \text{for each } k = 1, 2, \ldots n,$$

and then

$$-\sum_{h=1}^{k} \big(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\big) \prod_{s=1}^{h} \frac{(1+\sigma_s)}{(1+r_s)\big(1 - \frac{1}{\gamma_s \nu_s}\big)} \leq \frac{F_0}{W_0} \\ \text{for each } k = 1, 2, \ldots n. \tag{27}$$

3.1 Further conditions for the pension system sustainability

PROPOSITION 1 ***The sufficient condition for the sustainability.****Let the pension system have an initial non-negative fund, i.e. $F_0 \geq 0$.*
Sufficient condition for the pension system sustainability in $[0, n]$ *is that for each* $k = 1, 2, \dots n$ *contribution rate* α_k *is greater than or equal to the level of the unfunded contribution rate, i.e.*
if

$$\alpha_k \geq \alpha_k^{UN} = \frac{\beta_k}{\gamma_k \nu_k} \qquad \text{for each } k = 1, 2, \dots n \tag{28}$$

then

$$F_k \geq 0 \qquad \text{for each } k = 0, 1, \dots n.$$

Proof. This derives directly from Theorem 1. In fact, if (28) is true, then the l.h.s of condition (24) is surely less than or equal to zero, and therefore, as condition (24) is satisfied, the pension system is sustainable. □

For each $k \in \mathbb{N}$, $k \geq 1$, we define the following indicator.

DEFINITION 15 *The Logical Sustainability Indicator (LSI) of the pension system is denoted by* LSI_k *and is given by*

$$LSI_k = \frac{C_k \gamma_k \nu_k + F_k}{L_k^T}.$$

REMARK 2 Note that Proposition 1 can be expressed in terms of the LSI. Indeed, condition (28) in Proposition 1, by means of (15) and (12), can be written as

$$\alpha_k \geq \frac{1}{\gamma_k \nu_k} \frac{L_k^T - F_k}{W_k} \qquad \text{for each } k = 1, 2, \dots n$$

and hence equivalently

$$\frac{\alpha_k W_k \gamma_k \nu_k + F_k}{L_k^T} \geq 1 \qquad \text{for each } k = 1, 2, \dots n$$

and

$$\frac{C_k \gamma_k \nu_k + F_k}{L_k^T} \geq 1 \qquad \text{for each } k = 1, 2, \dots n.$$

Therefore, condition (28) in Proposition 1 is equivalent to

$$LSI_k \geq 1 \qquad \text{for each } k = 1, 2, \dots n.$$

It should be noted that the LSI analytical form is very similar to that of the Balance Ratio indicator used in the Swedish pension system. For the definition of the Balance Ratio and an exhaustive explanation of this indicator, refer to [14], whereas for further remarks on the comparison between these two indicators, see [2].

Remark 3 The sufficient condition for the sustainability can be expressed also referring to the indicators of the degree of PAYG covering of the pension disbursements and the degree of funding of the pension liability. Indeed, condition (28) in Proposition 1, by means of (16), can be also re-expressed as

$$\alpha_k \geq \frac{1}{\gamma_k \nu_k} \frac{L^T_{k-1}(1+r^L_k) - F_{k-1}(1+r_k)}{W_k} \qquad \textit{for each } k = 0, 1, 2, \ldots n$$

and hence equivalently

$$\frac{\alpha_k W_k \gamma_k \nu_k + F_{k-1}(1+r_k)}{L^T_{k-1}(1+r^L_k)} \geq 1 \qquad \textit{for each } k = 0, 1, 2, \ldots n.$$

Therefore, condition (28) in Proposition 1 is equivalent to

$$\frac{C_k}{P_k} + \frac{F_{k-1}(1+r_k)}{L^T_{k-1}(1+r^L_k)} \geq 1 \qquad \textit{for each } k = 0, 1, 2, \ldots n,$$

namely, by means of definitions (20) and (11),

$$Dc^{PAYG}_k + Dc^*_k \geq 1 \qquad \textit{for each } k = 1, 2, \ldots n.$$

3.2 The rule for the stabilization of the level of the unfunded pension liability in relation to wages

Let n be any time fixed, with $n \in \mathbb{N}$, $n \geq 1$. Following [2], we re-express some further relationships useful for controlling the pension system sustainability in the discrete context for time interval $[0, n]$, where upper limit n can also be equal to $+\infty$.

Proposition 2 ***The rule for the beta stabilization***.
Let us assume $0 \leq F_k \leq L^T_k$, *with* $L^T_k > 0$, *for a fixed* k, *with* $k = 0, 1, \ldots n-1$. *It is*

$$\Delta\beta_{k+1} = \beta_{k+1} - \beta_k = 0$$

if and only if

$$r^L_{k+1} = \frac{F_k}{L^T_k} r_{k+1} + \frac{L^T_k - F_k}{L^T_k} \sigma_{k+1} = Dc_k r_{k+1} + (1 - Dc_k)\sigma_{k+1}.$$

Proof. Note that from assumption $F_k \leq L^T_k$ it follows that $\beta_k \geq 0$.
We calculate the difference in the level of the unfunded pension liability in relation to wages between k and $k+1$

$$\Delta\beta_{k+1} = \beta_{k+1} - \beta_k = \frac{L^{UN}_{k+1}}{W_{k+1}} - \frac{L^{UN}_k}{W_k}. \tag{29}$$

By substituting in the previous formula the expressions of L_{k+1}^{UN} and W_{k+1}, given by (13) and (9) respectively, we obtain

$$\begin{aligned}\Delta\beta_{k+1} =& \frac{L_k^{UN} + L_k^T r_{k+1}^L - F_k r_{k+1} - L_k^{UN}(1+\sigma_{k+1})}{W_k(1+\sigma_{k+1})} = \\ =& \frac{L_k^T r_{k+1}^L - F_k r_{k+1} - L_k^{UN}\sigma_{k+1}}{W_k(1+\sigma_{k+1})}.\end{aligned} \tag{30}$$

Then $\Delta\beta_{k+1} = 0$ if and only if

$$r_{k+1}^L = \frac{F_k}{L_k^T} r_{k+1} + \frac{L_k^{UN}}{L_k^T}\sigma_{k+1} = \frac{F_k}{L_k^T} r_{k+1} + \frac{L_k^T - F_k}{L_k^T}\sigma_{k+1}.$$

□

REMARK 4 To fix the value of the beta indicator in time interval $[n_1, n_2]$, with $0 \leq n_1 < n_2 \leq n$, the rule for the beta stabilization has to be applied in all $(n_2 - n_1)$ unitary intervals included in interval $[n_1, n_2]$.

REMARK 5 Note that from relationship (29) it follows that

$$\Delta\beta_{k+1} = 0$$

if and only if

$$\frac{\Delta L_{k+1}^{UN}}{L_k^{UN}} = \frac{\Delta W_{k+1}}{W_k} = \sigma_{k+1}, \tag{31}$$

where it is $\Delta L_{k+1}^{UN} = L_{k+1}^{UN} - L_k^{UN}$ and $\Delta W_{k+1} = W_{k+1} - W_k$.

REMARK 6 Note that in the rule for the beta stabilization rates r_{k+1} and σ_{k+1} can be outcomes determined by stochastic processes.

PROPOSITION 3 ***The rule for the Dc_k stabilization.***
Let us assume $0 \leq F_k < L_k^T$ for a fixed k, with $k = 0, 1, \ldots n-1$. It is

$$\Delta Dc_{k+1} = Dc_{k+1} - Dc_k = 0$$

if and only if

$$\alpha_{k+1} = \frac{L_k^T}{W_{k+1}}\left(\frac{1 + r_{k+1}^L}{\gamma_{k+1}\nu_{k+1}} - \frac{F_k(r_{k+1} - r_{k+1}^L)}{\beta_k W_k}\right). \tag{32}$$

Proof. We calculate the difference in the degree of funding of the pension liability between time k and $k+1$, namely

$$\Delta Dc_{k+1} = Dc_{k+1} - Dc_k = \frac{F_{k+1}}{L_{k+1}^T} - \frac{F_k}{L_k^T}.$$

By means of the evolution equations of the fund and the pension liability, see (2) and (8), respectively, this difference can be expressed as

$$\begin{aligned}\Delta Dc_{k+1} &= \frac{(F_k(1+r_{k+1}) + C_{k+1} - P_{k+1})L_k^T}{L_{k+1}^T L_k^T} + \\ &- \frac{(L_k^T(1+r_{k+1}^L) + C_{k+1} - P_{k+1})F_k}{L_{k+1}^T L_k^T} = \\ &= \frac{F_k L_k^T(r_{k+1} - r_{k+1}^L) + (C_{k+1} - P_{k+1})(L_k^T - F_k)}{L_{k+1}^T L_k^T}.\end{aligned}$$

Then $\Delta Dc_{k+1} = 0$ if and only if

$$P_{k+1} - C_{k+1} = \frac{F_k L_k^T(r_{k+1} - r_{k+1}^L)}{L_k^T - F_k}. \tag{33}$$

Through algebraic calculation, using (1), (18), and (15), we obtain (32). □

Remark 7 As for the beta indicator, to fix the value of the degree of funding of the pension liability in time interval $[n_1, n_2]$, with $0 \leq n_1 < n_2 \leq n$, the rule provided by (32) has to be applied in all $(n_2 - n_1)$ unitary intervals included in interval $[n_1, n_2]$.

Remark 8 If $F_k = 0$, then the contribution rate provided by (32) equals the PAYG contribution rate at $k+1$, see (21). Note that if $F_k > 0$ and $r_{k+1} > r_{k+1}^L$, then it follows that $\alpha_{k+1} < \alpha_{k+1}^{PAYG}$, with α_{k+1} given by (32) .

Remark 9 Let us assume that $F_k > 0$ and that α_{k+1} is given by (32). By means of Proposition 3, it is $\Delta Dc_{k+1} = 0$, and in particular condition (33) holds. From this last one, we have that

$$\Delta F_{k+1} = F_{k+1} - F_k = F_k r_{k+1} - \frac{F_k L_k^T(r_{k+1} - r_{k+1}^L)}{L_k^T - F_k},$$

and hence

$$\Delta F_{k+1} = \frac{F_k}{L_k^T - F_k}\big(L_k^T r_{k+1}^L - F_k r_{k+1}\big). \tag{34}$$

Therefore, dividing both sides of (34) by F_k, and using (13), we have that

$$\Delta Dc_{k+1} = 0$$

if and only if

$$\frac{\Delta F_{k+1}}{F_k} = \frac{\Delta L_{k+1}^{UN}}{L_k^{UN}}.$$

Furthermore, as it can be easily proved that

$$\frac{\Delta\beta_{k+1}}{\beta_k} = \frac{W_k}{W_{k+1}}\left(\frac{\Delta L^{UN}_{k+1}}{L^{UN}_k} - \frac{\Delta W_{k+1}}{W_k}\right)$$

(see footnote [2]), then it follows also that

$$\Delta Dc_{k+1} = 0$$

if and only if

$$\frac{\Delta F_{k+1}}{F_k} = \frac{W_{k+1}}{W_k}\frac{\Delta\beta_{k+1}}{\beta_k} + \frac{\Delta W_{k+1}}{W_k}. \tag{35}$$

PROPOSITION 4 *Let us assume* $0 < F_0 < L^T_0$.
Furthermore, for each $k = 0, 1, \ldots, n-1$ *let us assume*

$$(A) \quad r^L_{k+1} = \frac{F_k}{L^T_k} r_{k+1} + \frac{L^T_k - F_k}{L^T_k}\sigma_{k+1}$$

$$(B) \quad \alpha_{k+1} = \frac{L^T_0}{W_0}\frac{1 + r^L_{k+1}}{(1+\sigma_{k+1})\gamma_{k+1}\nu_{k+1}} - \frac{F_0}{W_0}\frac{(r_{k+1} - \sigma_{k+1})}{(1+\sigma_{k+1})}$$

then it is $\beta_k = \beta_0$ *and* $Dc_k = Dc_0$ *for each* $k = 0, 1, ..., n$ *.*

Proof. From hypothesis (A) we have $\Delta\beta_{k+1} = 0$ for each $k = 0, 1, \ldots, n-1$, i.e. $\beta_k = \beta_0$ for each $k = 0, 1, \ldots, n$, see Proposition 2, the rule for the beta stabilization.

Let us consider difference $r_{k+1} - r^L_{k+1}$. Using hypothesis (A), we have that

$$r_{k+1} - r^L_{k+1} = \frac{L^T_k - F_k}{L^T_k}(r_{k+1} - \sigma_{k+1}) \qquad \text{for each } k = 0, 1 ... n-1,$$

and also, from (16),

$$\left(r_{k+1} - r^L_{k+1}\right) = \frac{\beta_k W_k}{L^T_k}(r_{k+1} - \sigma_{k+1}) \qquad \text{for each } k = 0, 1 ... n-1. \tag{36}$$

Proposition 3 establishes that $\Delta Dc_{k+1} = 0$ for each $k = 0, 1, \ldots, n-1$ if and only if condition (32) holds. Hence, using (36), we obtain that $\Delta Dc_{k+1} = 0$ for

[2] Indeed, it is

$$\begin{aligned}\frac{\Delta\beta_{k+1}}{\beta_k} &= \left(\frac{L^{UN}_{k+1}}{W_{k+1}} - \frac{L^{UN}_k}{W_k}\right)\frac{W_k}{L^{UN}_k} = \\ &= \left(\frac{L^{UN}_{k+1}W_k - L^{UN}_k W_{k+1} - L^{UN}_k W_k + L^{UN}_k W_k}{W_{k+1}W_k}\right)\frac{W_k}{L^{UN}_k} = \\ &= \left(\frac{W_k}{W_{k+1}}\frac{\Delta L^{UN}_{k+1}}{L^{UN}_k} - \frac{\Delta W_{k+1}}{W_{k+1}}\right) = \frac{W_k}{W_{k+1}}\left(\frac{\Delta L^{UN}_{k+1}}{L^{UN}_k} - \frac{\Delta W_{k+1}}{W_k}\right).\end{aligned}$$

each $k = 0, 1, \ldots, n-1$ if and only if

$$\begin{aligned}\alpha_{k+1} &= \frac{L_k^T}{W_{k+1}}\left(\frac{1+r_{k+1}^L}{\gamma_{k+1}\nu_{k+1}} - \frac{F_k}{L_k^T}(r_{k+1}-\sigma_{k+1})\right) = \\ &= \frac{L_k^T(1+r_{k+1}^L)}{W_{k+1}\gamma_{k+1}\nu_{k+1}} - \frac{F_k(r_{k+1}-\sigma_{k+1})}{W_{k+1}} = \\ &= \frac{L_k^T(1+r_{k+1}^L)}{W_k(1+\sigma_{k+1})\gamma_{k+1}\nu_{k+1}} - \frac{F_k(r_{k+1}-\sigma_{k+1})}{W_k(1+\sigma_{k+1})} \\ &\qquad \text{for each } k = 0, 1, \ldots, n-1.\end{aligned} \tag{37}$$

It is worth noting that $\Delta\beta_{k+1} = 0$ for each $k = 0, 1, \ldots, n-1$; then from condition (35) in Remark 9, it follows that

$$\frac{\Delta F_{k+1}}{F_k} = \frac{\Delta W_{k+1}}{W_k} \qquad \text{for each } k = 0, 1, \ldots, n-1, \tag{38}$$

and from condition (31) in Remark 5 it follows that

$$\frac{\Delta L_{k+1}^{UN}}{L_k^{UN}} = \frac{\Delta W_{k+1}}{W_k} \qquad \text{for each } k = 0, 1, \ldots, n-1. \tag{39}$$

From (38) and (39), we obtain respectively that

$$\frac{F_{k+1}}{W_{k+1}} = \frac{F_k}{W_k} \quad \text{and} \quad \frac{L_{k+1}^{UN}}{W_{k+1}} = \frac{L_k^{UN}}{W_k} \qquad \text{for each } k = 0, 1, \ldots, n-1, \tag{40}$$

and hence also

$$\frac{L_{k+1}^T}{W_{k+1}} = \frac{L_k^T}{W_k} \qquad \text{for each } k = 0, 1, \ldots, n-1. \tag{41}$$

Consequently, by substituting expressions (40) and (41) into expression (37), it follows that $\Delta Dc_{k+1} = 0$ for each $k = 0, 1 \ldots n-1$ if and only if

$$\alpha_{k+1} = \frac{L_0^T}{W_0}\frac{1+r_{k+1}^L}{(1+\sigma_{k+1})\gamma_{k+1}\nu_{k+1}} - \frac{F_0}{W_0}\frac{(r_{k+1}-\sigma_{k+1})}{(1+\sigma_{k+1})} \quad \text{for each } k = 0, 1 \ldots n-1.$$

□

4 The efficiency of the rule for the β_k stabilization

PROPOSITION 5 *Let us assume that* $0 \le F_0 \le L_0^T$. *Furthermore, let us assume that*

(a) the sequence of contribution rates is bounded above,
i.e. there exists $\alpha_{Max} > 0$ *such that* $\alpha_k \le \alpha_{Max}$, $\forall k \in \mathbb{N}$, $k \ge 1$;

(b) *the yearly rate of revaluation of the total pension liability in year* $(k+1)$ *follows rule*

$$r^L_{k+1} = \frac{F_k}{L^T_k} r_{k+1} + \frac{L^T_k - F_k}{L^T_k} \sigma_{k+1} + h_{k+1} \quad \forall k \in \mathbb{N} \tag{42}$$

where $\{h_{k+1}\}$ *is a bounded below sequence of positive numbers, i.e. there exists a positive constant* h^+ *such that* $0 < h^+ \leq h_{k+1}$ $\forall k \in \mathbb{N}$*;*

(c) *there exist positive constants,* Q_1*,* Q_2*, and* M*, such that*

$$Q_1 \leq A_{k+1} \quad \text{with } A_{k+1} = \prod_{s=1}^{k+1} \frac{1+\sigma_s}{(1+r_s)(1-\frac{1}{\gamma_s \nu_s})} \quad \forall k \in \mathbb{N}, \tag{43}$$

$$\gamma_{k+1}\nu_{k+1} \leq Q_2 \quad \forall k \in \mathbb{N}, \tag{44}$$

$$L^T_k \geq M \cdot W_{k+1} \quad \forall k \in \mathbb{N}. \tag{45}$$

Then the pension system is not sustainable, i.e. there exists a time $\bar{k}$*,* $\bar{k} \in \mathbb{N}$*, such that* $F_{\bar{k}} < 0$*.*

Proof. The proof is by contradiction, i.e. we assume that $F_k \geq 0$ $\forall k \in \mathbb{N}$. We show that this assumption leads to a contradiction.
Firstly, we consider the difference in the level of the unfunded pension liability in relation to wages between k and $k+1$, $\Delta\beta_{k+1}$, see (30), given by

$$\Delta\beta_{k+1} = \frac{L^T_k r^L_{k+1} - F_k r_{k+1} - L^{UN}_k \sigma_{k+1}}{W_{k+1}} \qquad \forall k \in \mathbb{N}. \tag{46}$$

From assumption (42) it is

$$L^T_k r^L_{k+1} = F_k r_{k+1} + (L^T_k - F_k)\sigma_{k+1} + L^T_k h_{k+1} \qquad \forall k \in \mathbb{N}.$$

Therefore, using the latter in (46), it follows that

$$\Delta\beta_{k+1} = \frac{L^T_k}{W_{k+1}} h_{k+1} \qquad \forall k \in \mathbb{N}.$$

As $\{h_{k+1}\}$ is a bounded below sequence of positive numbers, and using assumption (45), then it follows that

$$\Delta\beta_{k+1} = \frac{L^T_k}{W_{k+1}} h_{k+1} \geq M h^+ > 0 \qquad \forall k \in \mathbb{N}. \tag{47}$$

Hence, from (47), it follows that sequence β_k is monotonically increasing.
We have assumed, by contradiction, that $F_k \geq 0$ $\forall k \in \mathbb{N}$. We prove that this assumption leads to a contradiction, from which the thesis follows.
According to the necessary and sufficient condition for the sustainability, refer

to condition (27) in Remark 1, under assumption $F_0 \geq 0$, the pension system is sustainable in $[0, +\infty)$, namely $F_k \geq 0$ $\forall k \in \mathbb{N}$, $k \geq 1$, if and only if it results

$$-\sum_{h=1}^{k}(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h})\prod_{s=1}^{h}\frac{1+\sigma_s}{(1+r_s)(1-\frac{1}{\gamma_s \nu_s})} \leq \frac{F_0}{W_0} \qquad \forall k \in \mathbb{N}, k \geq 1.$$

Equivalently, $F_k \geq 0$ $\forall k \in \mathbb{N}$, $k \geq 1$, if and only if it results

$$\sum_{h=1}^{k}\big(\frac{\beta_h}{\gamma_h \nu_h} - \alpha_h\big)A_h \leq \frac{F_0}{W_0} \qquad \forall k \in \mathbb{N}, k \geq 1, \tag{48}$$

where the notation introduced in assumption (c) is used.
By assumption (c), see (43) and (44), and by assumption (a), it follows

$$Q_1 \sum_{h=1}^{k}\big(\frac{\beta_h}{Q_2} - \alpha_{Max}\big) \leq \sum_{h=1}^{k}\big(\frac{\beta_h}{\gamma_h \nu_h} - \alpha_h\big)A_h \qquad \forall k \in \mathbb{N}, k \geq 1. \tag{49}$$

Hence, from (48) and (49) it must result that

$$Q_1 \sum_{h=1}^{k}\big(\frac{\beta_h}{Q_2} - \alpha_{Max}\big) \leq \frac{F_0}{W_0} \qquad \forall k \in \mathbb{N}, k \geq 1; \tag{50}$$

but this last one, (50), cannot be true $\forall k \in \mathbb{N}$, $k \geq 1$. Indeed, as sequence $\{\beta_k\}$ has been proved to be positively divergent, quantity $\sum_{h=1}^{k}\big(\frac{\beta_h}{Q_2} - \alpha_{Max}\big)$ at l.h.s of (50) approaches to $+\infty$ for k approaching to $+\infty$. Hence, there exists a $\bar{k}$, $\bar{k} \in \mathbb{N}$, such that $F_{\bar{k}} < 0$. By contradiction the theorem thesis follows. □

Remark 10 Note that if there exists a positive constant, K^*, such that

$$r_s - \sigma_s - \frac{1}{\gamma_s \nu_s} > K^* > 0 \qquad \forall s \in \mathbb{N}, s \geq 1,$$

then sequence A_k approaches to zero for k approaching to $+\infty$, and hence assumption (43) does not hold.

Proposition 6 *Let us assume that* $0 \leq F_0 \leq L_0^T$. *Furthermore, let us assume that*

(a) the yearly rate of revaluation of the total pension liability in year $(k+1)$ *follows rule*

$$r_{k+1}^L = \frac{F_k}{L_k^T} r_{k+1} + \frac{L_k^T - F_k}{L_k^T}\sigma_{k+1} - h_{k+1} \quad \forall k \in \mathbb{N} \tag{51}$$

where $\{h_{k+1}\}$ *is a bounded below sequence of positive numbers, i.e. there exists a positive constant* h^- *such that* $0 < h^- \leq h_{k+1}$, $\forall k \in \mathbb{N}$;

(b) there exist positive constants M and R such that

$$M \cdot W_{k+1} \leq L_k^T \leq RW_k \quad \forall k \in \mathbb{N}. \tag{52}$$

Then sequence $\{D_{ck}\}$ is positively divergent.

Proof. Analogously to the proof of previous Proposition 5, we consider difference $\Delta\beta_{k+1}$ that, by (30), is given by

$$\Delta\beta_{k+1} = \frac{L_k^T r_{k+1}^L - F_k r_{k+1} - L_k^{UN}\sigma_{k+1}}{W_{k+1}} \qquad \forall k \in \mathbb{N}. \tag{53}$$

From assumption (51) it is

$$L_k^T r_{k+1}^L = F_k r_{k+1} + (L_k^T - F_k)\sigma_{k+1} - L_k^T h_{k+1} \qquad \forall k \in \mathbb{N},$$

and substituting in (53), it follows that

$$\Delta\beta_{k+1} = -\frac{L_k^T}{W_{k+1}} h_{k+1} \qquad \forall k \in \mathbb{N}.$$

Under the assumption that $\{h_{k+1}\}$ is a bounded below sequence of positive numbers, and using assumption (52), then it follows that

$$\Delta\beta_{k+1} = -\frac{L_k^T}{W_{k+1}} h_{k+1} \leq -Mh^- < 0 \qquad \forall k \in \mathbb{N}. \tag{54}$$

Therefore, sequence $\{\beta_k\}$ is monotonically decreasing, and also negatively divergent. This means that the fund is sistematically greater than the pension liability, that is the pension system is overcapitalized.
Using relationship (17), we can express D_{ck} as

$$D_{ck} = 1 - \beta_k \frac{W_k}{L_k^T} \qquad \forall k \in \mathbb{N}.$$

As sequence $\{\beta_k\}$ is proved to be negatively divergent, and it is $\frac{W_k}{L_k^T} \geq \frac{1}{R}$ by assumption (52), then sequence D_{ck} is positively divergent. □

5 Conclusions

In this paper we provide the model of the logical sustainability of defined contribution pension systems in the discrete framework (for the continuous framework, see [2]) under the general assumptions of variable mortality and stochastic financial rate of the pension system fund and stochastic productivity of the active participants. These assumptions must not be neglected, as usually occurs, particularly when the pension systems are PAYG financed. As a matter of fact, under these realistic assumptions, the model proposed allows to hedge the sustainability in a logical mathematical key. This means that in this realistic framework specific conditions for the sustainability are proved. Hence, the sustainability is not based on actuarial projections. Therefore, the sustainability we propose is "strong" as opposite to the "weak" sustainability that is based on actuarial projections and, hence, based on the underlying assumptions.

We propose an appropriate choice of the rate of return on the pension liability, which takes into account the financial return on the pension system fund and the productivity of the active participants. This rule allows to stabilize the beta indicator, that is the ratio between the unfunded pension liability and the wages, and is linked to the minimum contribution rate for the system sustainability. Furthermore, it is applicable out of the steady state, and whichever the actual trends of the financial rate and the productivity rate could be. Note that the problem of how to measure the rate of return to a no-steady state PAYG pension system is one of the major issues to consider in the academic literature and in the actuarial practice, [15].

In addition, in this paper we show that the proposed rule on the rate of return on the pension liability is also the efficient one: that is, if the rate of return on the pension liability is higher or lower than that provided by the rule, then the pension becomes unsustainable or overcapitalized, respectively.

Our further goal is to transform from the continuous to the discrete framework also the part of the logical sustainability model related to phenomena of economic/demographic unbalances (referred to as economic/demographic waves). In our opinion, the model of the logical sustainability of pension systems is general and actually tractable, and is able to address other concrete issues such as the adequacy of the pension benefits or the gender inequality at retirement.

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