Local Estimation of Stock Market Efficiency

SERGIO BIANCHI sbianchi@eco.unicas.it
ALEXANDRE PANTANELLA a.pantanella@eco.unicas.it
AUGUSTO PIANESE pianese@unicas.it

University of Cassino
Department of Economics and L.I.S.A.
Campus Folcara, Via S. Angelo - 03043 Cassino
ITALY

Abstract: A dynamical measure is designed to assess the time-changing degree of efficiency of stock markets, under the hypothesis that the price can be modeled by the Multifractional Processes with Random Exponent (MPRE), a class of stochastic processes defined to make the fractional Brownian motion more versatile in describing nonhomogeneous dynamics. Our findings show that efficiency appears at low frequencies as a consequence of the balancement of local inefficiencies of opposite sign.

Key–Words: Multifractional Processes with Random Exponent, Hölder exponent, Hurst parameter estimation, Efficient Market Hypothesis

1 Introduction

Maybe the most critical issue on which mathematical finance relies, concerns the unsolved problem of market efficiency. Since the Eugene Fama's influential article [10], it was widely accepted that securities markets were efficient: prices of individual stocks at time t, p_t, were expected to discount all the information up to time t, $\mathscr{F}_t$, as a consequence of the quick spread of the news which should ensure that eventual deviations from equilibrium values could not last for long.

Formally, this means that the one-period excess return $z_{t+1} = r_{t+1} - \mathbb{E}(r_{t+1}|\mathscr{F}_t)$ (where $r_{t+1} = \frac{p_{t+1}-p_t}{p_t}$) is

$$\mathbb{E}(z_{t+1}|\mathscr{F}_t) = 0 \quad (1)$$

that is a fair game with respect to $\mathscr{F}_t$[(i)]. In a perfectly efficient market, profitable arbitrage are excluded as prices would adjust instantaneously to new information. Differently, if the market is not fully efficient, arbitrage opportunities arise. In the last twenty years, the Efficient Market Hypothesis was weakened in a number of ways (see [16] for a survey), to the extent that a strand of skeptical thought has been growing more and more, along with the attempts to conciliate the EMH with other psychology-driven approaches. In this regard, several recent contributions have started this research topic: among them, the new paradigm of Adaptive Market Hypothesis (AMH), proposed by Lo [11], tries to agree in a coherent frame the EMH and the Beahvioural Finance. The same goal is pursued by [7] and [8], works in which the benefits are analyzed to model the financial dynamics by multifractional processes.

In the wake of these attempts, we discuss a stochastic model, recently introduced by Ayache and Taqqu [1] in a general setting (i.e. without any specific concern to financial analysis), which in a parsimonious way can conciliate both efficiency and inefficiency. We show that the estimation of a key-parameter of this model allows to assess the local degree of efficiency and to distinguish between what we name *positive* and *negative* inefficiency.

In addition, the empirical evidence supplied in this work shows that, in fact, the overall efficiency of stock markets does result from the alternation of inefficiencies of opposite signs (what the Behavioral Finance would classify as underreaction and overreaction).

The paper is organized as follows: in Section 2, the Multifractional Processes are recalled and a particular

[(i)] In this respect, it is worthwhile to recall what Fama notices in his seminal work [10]:

> But we should note right off that, simple as it is, the assumption that the conditions of market equilibrium can be stated in terms of expected returns elevates the purely mathematical concept of expected value to a status not necessarily implied by the general notion of market efficiency. The expected value is just one of many possible summary measures of a distribution of returns, and market efficiency per se (i.e., the general notion that prices "fully reflect" available information) does not imbue it with any special importance

ISBN: 978-1-61804-064-0

attention is paied to the MPRE, that is the model we assume to describe financial prices (subsection 2.2). Section 3 describes how an estimation of the pointwise Hölder exponent can be obtained using an estimator recently improved by the authors. In Section 4 the dynamical assessment of the local efficiency is discussed. Section 5 is devoted to the empirical application to three stock indexes and, finally, Section 6 concludes.

2 The Multifractional Processes

The analysis we performed relies on the assumption that the price process can be modelled by the Multifractional Processes with Random Exponent (MPRE), a very versatile class of stochastic processes introduced by [1]. The basics of MPRE can be recalled by comparison with the multifractional Brownian motion (mBm), process with whom the MPRE itself shares some properties. In its turn the mBm is a generalisation of the celebrated fractional Brownian motion (fBm) (albeit the semantic poverty, one should avoid confusing the multifractional processes with the multifractal processes (see [3], [5]).

2.1 fBm and mBm

The fractional Brownian motion (fBm), denoted by $B_H(t)$, is a centered Gaussian, self-similar, continuous stochastic process with almost surely non differentiable sample paths and stationary increments. It generalizes the Brownian motion replacing the value $\frac{1}{2}$ by the Hurst exponent $H \in (0,1)$ [12] and admits the non anticipative moving average representation

$$B_H(t) = KV_H \int_{\mathbb{R}} \left((t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}} \right) dW(s) \qquad (2)$$

where $x_+ = \max(x,0)$, $V_H = \frac{\Gamma(2H+1)\sin(\pi H)^{\frac{1}{2}}}{\Gamma\left(H+\frac{1}{2}\right)}$ is a normalizing factor, $K^2 = Var B_H(1)$ and, as usual, dW denotes the Brownian measure. Up to a multiplicative constant representation (2) is equivalent to the following harmonizable one

$$\bar{B}_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{i\xi\,|\xi|^{H-\frac{1}{2}}} d\hat{W}(\xi) \qquad (3)$$

$d\hat{W}$ being the Fourier transform of the Brownian measure dW, i.e. unique complex-valued stochastic measure such that $\int_{\mathbb{R}} f(x)dW(x) = \int_{\mathbb{R}} \hat{f}(x)d\hat{W}(x)$ for all $f \in L^2(\mathbb{R})$, where $\hat{f}(x) = \int_{\mathbb{R}} e^{-i\xi x} f(x)dx$ is the Fourier transform of f. Another useful representation of fBm uses wavelets; in this case the standard random wavelet series representation reads as

$$\hat{B}(x,H) = \sum_{j=-\infty}^{+\infty} \sum_{k\in\mathbb{Z}} 2^{-jH} \epsilon_{j,k} \left(\Psi(2^j x - k, H) - \right.$$
$$\left. -\Psi(-k,H) \right) \qquad (4)$$

where:

- $\{\epsilon_{j,k}\}_{(j,k)\in\mathbb{Z}^2}$ is a sequence of independent $N(0,1)$ random variables;
- $\Psi \in C^\infty(\mathbb{R}\times(0,1))$ is well-localized in the first variable and uniformly localized in H, which means that, for all $(n,p) \in \mathbb{N}^2$,

$$\sup\left\{ (1+|x|)^p |(\partial_x^{(n)}\Psi)(x,H)| : (x,H) \in \mathbb{R}\times(0,1) \right\} < +\infty.$$

The random wavelet series representation of fBm is, with probability 1, uniformly convergent in (x,H) on each compact subset of $\mathbb{R}\times(0,1)$. This result will be useful in the sequel.

The increments of $B_H(t)$ have autocovariance function explicitly given by

$$\rho_H(k) = \frac{K^2}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H} \right) = $$
$$= \frac{K^2}{2}\Delta^2 |k|^{2H} \qquad (5)$$

where the symbol Δ^2 denotes the second difference. From (5) it trivially follows that the increments of fBm display long-range dependence when $H > \frac{1}{2}$ as, in this case, it is $\sum_{k\in\mathbb{Z}} |\rho_H(k)| = +\infty$.

It is worthwhile to recall also that the pointwise Hölder exponent of the fBm is $\alpha_{B_H}(t) = H$, almost surely at any point t, and the same occurs for its uniform Hölder exponent over an arbitrary non-degenerate interval J, $\beta_{B_H}(J) = H$[(ii)].Just the constancy of the Hölder regularity seems too restrictive

[(ii)] Remind, see [1], that the pointwise Hölder exponent of a process $X(t,\omega)$ seizes the regularity of the process graph. In details, given the stochastic process $X(t,\omega)$ with almost surely continuous and non differentiable paths, the pointwise Hölder exponent at t is defined as

$$\alpha_X(t,\omega) = \sup\left\{ \alpha : \limsup_{h\to 0} \frac{|X(t+h,\omega) - X(t,\omega)|}{|h|\alpha} = 0 \right\}.$$

Similarly, the uniform Hölder exponent over a non-degenerate interval J measures the global Hölder regularity over J; it is defined as the random variable

$$\beta_X(j,\omega) = \sup\left\{ \beta : \sup_{t,t'\in J} \frac{|X(t',\omega) - X(t,\omega)|}{|t'-t|^\beta} < \infty \right\}.$$

It is always $\beta_X(J,\omega) \le \inf_{t\in J} \alpha_X(t,\omega)$.

ISBN: 978-1-61804-064-0

to depict the complexity of financial dynamics, characterized by volatility clustering.

The multifractional Brownian motion (mBm), independently introduced by [14] and [2], overcomes this problem, by replacing the constant parameter H by the Hölderian deterministic function $h(t)$[(iii)]. The non anticipative representation is similar to (2) and reads as

$$X_{h(t)}(t) = KV_{h(t)} \int_{\mathbb{R}} \left((t-s)_+^{h(t)-\frac{1}{2}} - \\ -(-s)_+^{h(t)-\frac{1}{2}} \right) dW(s). \tag{6}$$

Using the notation as in the case of fBm, the harmonizable representation of mBm is similar to (3)

$$\bar{X}_{h(t)}(t) = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{i\xi\,|\xi|^{h(t)-\frac{1}{2}}} d\hat{W}(\xi) \tag{7}$$

but, in this case, the distributions of $X_{h(t)}(t)$ and $\bar{X}_{h(t)}(t)$ are not properly the same but just nearly the same, as pointed out by [15].

The Hölderianity of the function h implies that, almost surely, $\alpha_{X_{h(t)}(t)} = h(t)$ and $\beta_{X_{h(t)}(t)} = \inf_{t\in J} h(t)$, for any non-degenerate interval $J \subset [0,1]$. In other words, the pointwise and the uniform Hölder exponents of the mBm equal $h(t)$, with probability 1.

This result makes the model more versatile and able to capture the time-changing volatility of actual financial markets.

An important property of the mBm, which will be exploited in the following, states that for each $u \in \mathbb{R}$

$$\lim_{a\to 0^+} \frac{X_{h(t+au)}(t+au) - X_{h(t)}(t)}{a^{h(t)}} \stackrel{d}{=} B_{H(t)}(u) \tag{8}$$

namely that at any point t there exists an fBm of parameter $H(t)$ tangent to the mBm (see [2]).

2.2 Multifractional Processes with Random Exponents

For practical modelling purposes, the limit of the mBm resides in the deterministic nature of the function $h(t)$. In particular, when one refers to financial markets it seems reasonable to assume nondeterministic functional parameters (see, e.g., [5], [6] or [9] for a discussion of this issue). Even if without any specific concern to practical applications, Ayache and Taqqu [1] addressed the problem of defining a new process in which the parameter H of fBm is replaced by a stochastic process[(iv)] $\{S(t,\omega)\}_{t\in\mathbb{R}}$ with values in the fixed interval $[a,b] \subset (0,1)$. As shown by Papanicolaou and Sølna [13], the stochastic integral

$$Z(t,\omega) = \int_{\mathbb{R}} \frac{e^{it\xi}-1}{i\xi\,|\xi|^{S(t)-\frac{1}{2}}} d\hat{W}(\xi) \tag{9}$$

is well-defined when the process $\{S(t,\omega)\}$ is independent of $d\hat{W}$ and, in this case, the main results stated on mBm apply to $\{Z(t,\omega)\}_{t\in\mathbb{R}}$. Since (9) is no longer defined when $\{S(t,\omega)\}$ depends on $d\hat{W}$, Ayache and Taqqu consider the standard random wavelet series representation of fBm (4) and, exploiting its almost sure uniform convergence in (x,H) on each compact subset of $\mathbb{R} \times (0,1)$, replace (x,H) by $(t, S(t,\omega))$, so defining the non Gaussian process they named Multifractional Process with Random Exponent (MPRE). Their construction considers:

- the stochastic process $S : t \in [0,1] \to [a,b] \subset [0,1]$ (without loss of generality one could replace the time domain $[0,1]$ by any compact interval);
- the Gaussian field $B_H(t)_{t,H} \in [0,1] \times [a,b] \subset (0,1)$ described by (2) and (3) (with the clarification that the term 'field' refers to the fact that both t and H changes);

With these premises, the Multifractional Process with Random Exponent (MPRE) of parameter $S(t,\omega)$ is defined as

$$Z(t,\omega) = f_2(f_1(t)) = B_{S(t,\omega)}(t,\omega) \tag{10}$$

where $f_1 : [0,1] \to [0,1] \times [a,b]$ (i.e., $t \mapsto (t, S(t,\omega))$) and $f_2 : [0,1]\times[a,b] \to \mathbb{R}$ (i.e., $(t,H) \mapsto B_H(t,\omega)$).

Notice that the construction of the MPRE does not necessarily require neither stationarity nor independence of $S(t,\omega)$ on the Brownian motion W. When independence is assumed, the MPRE recovers the main results stated for the mBm, whereas when dependence is postulated, the kernel $(t-s)_+^{S(t,\omega)-1/2} - (-s)_+^{S(t,\omega)-1/2}$ is no longer adapted to the natural filtration of W and the integral is no longer defined. Anyway, this does not impede from using, even in this case, the standard random wavelet series representation (4), which does not involve the variable s.

It is worthwhile to recall four relevant features of the MPRE proved by [1]:

- the continuity of the paths of $\{S(t,\omega)\}$ implies the continuity of $\{Z(t,\omega)\}$. In addition, if S is a non-degenerate process, $Z(t,\omega)$ is not Gaussian;

[(iii)] Remind that the function $h(t)$ is Hölderian of order β on each compact interval $J \subset \mathbb{R}$ if, for each $t,s \in J$ and for $c > 0$, it holds $|h(t)-h(s)| \le c|t-s|^{\beta}$, where $\beta > \max_{t\in J} h(t)$.

[(iv)] To avoid ambiguity, when necessary, we write explicitly ω for the stochastic process.

ISBN: 978-1-61804-064-0

- if $\beta_S([0,1]) > \sup_{t\in[0,1]} S(t,\omega)$ with probability 1, namely the uniform Hölder exponent is larger than the supremum of stochastic parameter $S(t,\omega)$, then almost surely $\alpha_Z(t,\omega) = S(t,\omega)$ at any point $t \in (0,1)$ and $\beta_Z(J,\omega) = \inf_{t\in J} S(t,\omega)$, for any non-degenerate interval $J \subset [0,1]$.
  This means that, almost surely, the pointwise Hölder exponent of the MPRE equals its stochastic functional parameter S and the uniform Hölder exponent equals the infimum of S over J, so preserving the information in terms of pointwise regularity of the process;
- if S is a random variable independent of W in (9), then the increments of Z form a stationary sequence, that is – denoting by $\stackrel{d}{=}$ the equality of finite-dimensional distributions – it holds

$$\{(Z(t+h) - Z(t)\} \stackrel{d}{=} \{(Z(h) - Z(0)\}$$

  for $h \in [0, 1-t]$;
- if S is a stationary stochastic process independent of W in (9), then $\{Z(t,\omega)\}$ is self-similar[(v)] in its marginal distributions, namely $Z(at) \stackrel{d_1}{=} a^{S(t)} Z(t)$.

3 Estimation of the Hölder Exponent

From a practical viewpoint, given the process Z, the problem resides in getting an estimate of its stochastic parameter $S(t)$. Although many estimators have been proved to work for the constant parameter H of the fBm, an estimator of the stochastic paramenter $S(t,\omega)$ of MPRE is not available yet. Anyway, by exploiting the properties that MPRE shares with mBm when S is independent of W (see previous Section), one can get an estimation by using the class of *Moving Window Absolute Moment-Based Estimators (MWAM)* introduced in [4] and improved in [9]. As a consequence of relation (8), in a neighborhood of each point t_0 along the process trajectory, $Z(t,\omega)$ behaves like an fBm of parameter $H(t_0)$; MWAM estimator exploits this property and in detail the fact that locally, i.e. in a window of sufficiently small lenght, the random variables are expected to be normally distributed. This allows to deduce the class of estimators starting from the k^{th} absolute moment of a Gaussian random variable. In short, given the trajectory $Z(t) = Z(t, \omega = \bar{\omega})$ of the process, sampled at times $t = \frac{1}{N}, ..., \frac{N-1}{N}, 1$, let:

- q be the lag of the increments of Z;
- K^2 be the unit time variance of Z;
- k be the order of the absolute variation;
- δ be the length of the estimation window.

The MWAM estimator is defined as

$$\hat{S}^k_{\delta,q,N,K}(t) = \frac{\log\left(\sqrt{\pi}\,\mathbb{S}^k_{\delta,q,N}(t) / \left(2^{k/2}\Gamma\left(\frac{k+1}{2}\right)K^k\right)\right)}{k \log\left(\frac{q}{N-1}\right)} \tag{11}$$

where

$$\mathbb{S}^k_{\delta,q,N}(t) = \frac{\sum_{j=t-\delta}^{t-q} |Z(t+q) - Z(t)|^k}{\delta - q + 1}.$$

The MWAM estimator has a rate of convergence of $o(\delta^{-1/2}\log^- 1N)$ (see [3]), which allows to get reliable estimates even with very small windows.
What is relevant here is that the estimator is normally distributed with mean equal to $S(t)$. Moreover, when the parameter is constant and equals $\frac{1}{2}$, its variance – for $q = 1$ and $K = 1$ – can be explicitly calculated and is given by

$$Var\left(\hat{S}^k_{\delta,1,N,1}(t)\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{2k+1}{2}\right) - \Gamma^2\left(\frac{k+1}{2}\right)}{\delta k^2 \log^2(N-1)\Gamma^2\left(\frac{k+1}{2}\right)} \tag{12}$$

Notice that, even if the variance is calculated with respect to the case $K = 1$, relation (12) holds in general, since different values of K only translate the estimates, as pointed out in [9].

4 Assessing efficiency through the distribution of MWAM estimators

The idea founding this work is to use the distribution of the MWAM class of estimators to assess dynamically the degree of efficiency of stock markets.
Since market efficiency surely holds when $S(t) = \frac{1}{2}$, the natural way to evaluate the state of the market is to compare the estimated $\hat{S}^k_{\delta,q,N,K}(t)$ with this threshold. This can be done by exploiting relation (12), which provides the variance of the normally distributed estimates around $\frac{1}{2}$, namely the variance of the distribution

$$\Phi_{\left(\hat{S}^k_{\delta,q,N,K}|S(t)=\frac{1}{2}\right)}(z) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{(x-1/2)^2}{2\sigma^2}} dx \tag{13}$$

σ being the square root of (12). Notice that the distribution remains unchanged through time under the hypothesis of a constant $S(t) = \frac{1}{2}$.
On the other side, the empirical distribution is generally allowed to change through time. So, once a second window of lenght Δ has been fixed, one can slide

(v)Remind that the stochastic process $\{X(t,\omega)\}_{t\in T}$ is said self-similar of parameter H if for any $a > 0$ and $t \in T$, it is $\{X(at,\omega)\} \stackrel{d}{=} a^H \{X(t,\omega)\}$, where $\stackrel{d}{=}$ denote the equality of the finite-dimensional distributions of $\{X(t,\omega)\}$.

ISBN: 978-1-61804-064-0

the whole sample made of $N-\delta+1$[vi] observations, by estimating in each subinterval the empirical distribution

$$\hat{\Phi}_{\Delta,\tau}(z)=\frac{1}{\Delta}\sum_{i=1}^{\Delta}\mathbb{I}\left\{\hat{S}^{k}_{\delta,q,N,K}\left(\tau-\frac{\Delta}{2}+i\right)\leq z\right\}, \quad (14)$$

where $\tau=\frac{\Delta}{2},\ldots,N-\delta+1-\frac{\Delta}{2}$. Once the significant level α has been fixed, the values

$$z_1=\Phi^{-1}_{\left(\hat{S}^{k}_{\delta,q,N,K}|S(t)=\frac{1}{2}\right)}\left(\tfrac{\alpha}{2}\right), \quad \text{and}$$

$$z_2=\Phi^{-1}_{\left(\hat{S}^{k}_{\delta,q,N,K}|S(t)=\frac{1}{2}\right)}\left(1-\tfrac{\alpha}{2}\right)$$

can be calculated and, by the empirical distribution $\hat{\Phi}_{\Delta,\tau}$, the quantity

$$\mathscr{E}_{\Delta,\tau}(\alpha)=\hat{\Phi}_{\Delta,\tau}(z_2)-\hat{\Phi}_{\Delta,\tau}(z_1) \quad (15)$$

is expected to be close to the confidence level $(1-\alpha)$ if $S(t)=\frac{1}{2}$, that is if the market is efficient in the window $\left]\tau-\frac{\Delta}{2},\tau+\frac{\Delta}{2}\right]$. We assume $\mathscr{E}_{\Delta,\tau}(\alpha)$ as a measure of the *local efficiency*. Obviously, by construction, $\mathscr{E}_{\Delta,\tau}(\alpha)$ satisfies the σ-additivity property.
A separate and more cogent signal of efficiency would require $\hat{\Phi}_{\Delta,\tau}(z)$ to be Gaussian with mean $S(t)=\frac{1}{2}$. Obviously, this condition would imply $\mathscr{E}_{\Delta,\tau}(\alpha)$ to be (statistically) equal to $1-\alpha$ if z was continuous. Anyway, since in the application z is discrete-valued, the implication does not hold and it is significant to evaluate the two measures separately.
Both the tests will be performed in the next Section. In addition, it will be checked whether the distribution of $\hat{S}^{k}_{\delta,q,N,K}(t)$ is normal even when it is far from $\frac{1}{2}$, as it is expected to be when the process is a genuine mBm or, under particular conditions, an MPRE.

5 Empirical Application

We estimated the local efficiency for three main stock indexes, from January 1^{st}, 1998 to December 31^{st}, 2010. The dataset is summarized in Table 1.
The analysis was performed through these steps:

Ticker	Index	Initial date	End date	Obs.
DJI	Dow Jones Ind. Av.	19980101	20101231	3,271
N225	NIKKEI 225	19980101	20101231	3,190
FTSE	FTSE 100	19980101	20101231	3,283

Table 1: Dataset

[vi] Remind that of the initial N data, δ were used to calculate $\hat{S}^{k}_{\delta,q,N,K}(t)$

- the estimates $\hat{S}^{k}_{\delta,q,N,K}(t)$ were calculated setting $\delta=30$, $q=1$, $k=2$ (for a detailed discussion of these running parameters see [4] and [9]). Here we will just recall that the value $k=2$ ensures the minimal variance of the estimator;
- the empirical distributions $\hat{\Phi}_{\Delta,\tau}(z)$ of the estimates were calculated for each sliding window of length $\Delta=125$ trading days (about six trading months). The significance levels, needed to calculate the quantiles z_1 and z_2, were set to $\alpha=10\%, 5\%$ and 1%.
- along with the measure $\mathscr{E}_{\Delta,\tau}(\alpha)$, the D'Agostino K^2 test was run for each empirical distribution $\hat{\Phi}_{\Delta,\tau}(z)$ and – when the null hypothesis is not rejected – the Z test was performed to assess whether the population mean is $\frac{1}{2}$.

The results of our analysis are summarized in Figures 1, 2 and 3 and in Tables 2 and 3.
Panels (a) simply display the stock indexes through time.
Panels (b) display, for each index, the paths of the estimated stochastic parameter $S(t)$. It is worthwhile to note that for the overall time series, $\hat{S}^{k}_{\delta,q,N,K}(t)$ roughly fluctuates around the efficiency threshold $\frac{1}{2}$, as shown by Table 2, which summarizes the main distributional parameters of the estimates. In all the cases, even if large deviations take place, the overall mean is decidedly close to $\frac{1}{2}$ and this suggests that the two opposite sources of inefficiency ($S(t)>\frac{1}{2}$ and $S(t)<\frac{1}{2}$) tend to balance, when sufficiently long time spans are considered.
Panels (c) display the estimates of local efficiency

	DJIA	N225	FTSE
Mean	0.5274	0.4971	0.4745
Max	0.6483	0.6195	0.5920
Min	0.3469	0.3175	0.3007
Std	0.0523	0.0429	0.0538
Kurt	3.5823	4.9433	2.9587
Skew	-0.5573	-0.4115	-0.3678

Table 2: Main distributional parameters of $\hat{S}^{k}_{\delta,q,N,K}(t)$

	DJIA	N225	FTSE
% $N(\mu,\sigma^2)$	0.5189	0.5232	0.5884
% $N(\frac{1}{2},\sigma^2)$	(0.0032)	(0.0137)	(0.0168)

Table 3: Frequencies of acceptance of normality ($\alpha=0.05$)

for the three significance levels ($\alpha = 10\%, 5\%$ and 1%), along with the normal distributed estimates (green circles) and the normal distributed estimates with mean $\frac{1}{2}$ (red circles). It seems relevant that, according to our estimates, since 2004 the U.S. market was extremely inefficient with $S(t)$ *almost everywhere* above $\frac{1}{2}$. This reading suggests that the financial crisis of 2007-2009 could possibly have arbitraged away this inefficiency, at least till 2008 (first period of the crisis). Interestingly and counterintuitively, this extremely long inefficient period seems to be typical of the sole Dow Jones: Footsie 100 and Nikkei 225 show higher degrees of efficiency in the same time span. Also notice that, as stated by theory, the estimator is often normally distributed, even if almost never with mean $\frac{1}{2}$ (Table 3 displays the frequencies of acceptance of normality and – in parentheses – the same frequencies with respect to $\frac{1}{2}$).

6 Conclusions

A measure of the local efficiency of stock markets was defined in terms of the distribution of the estimated pointwise Hölder regularity of the paths designed by the price process. The measure relies on the assumption that the dynamics of stock prices can be modeled by a *Multifractional Process with Random Exponent*, a very versatile stochastic process recently defined to model time nonhomogeneous phenomena. The model appears as general as possible and balances its complexity with the capability it shows to seize many of the features displayed by actual financial data.

The measure we adopt distinguishes between *positive* and *negative* efficiency: the former occurs when $S(t) > \frac{1}{2}$ and the latter when $S(t) < \frac{1}{2}$. In this respect, future insights are likely come from a more close examination of the link between the MPRE model and the euristics of Behavioral Finance.

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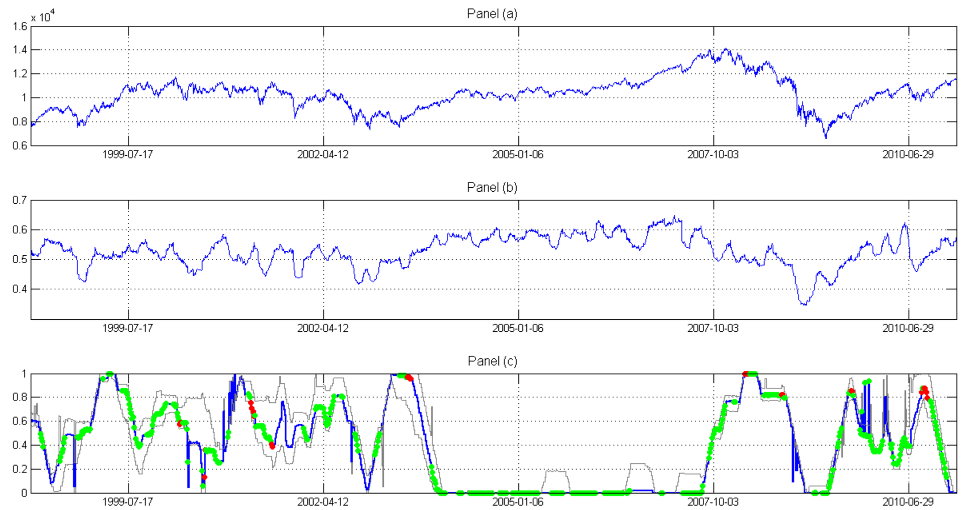

Figure 1: Dow Jones Industrial Average: (a) Price - (b) Estimated $S(t)$ - (c) $\mathscr{E}_{\Delta,\tau}(\alpha)$

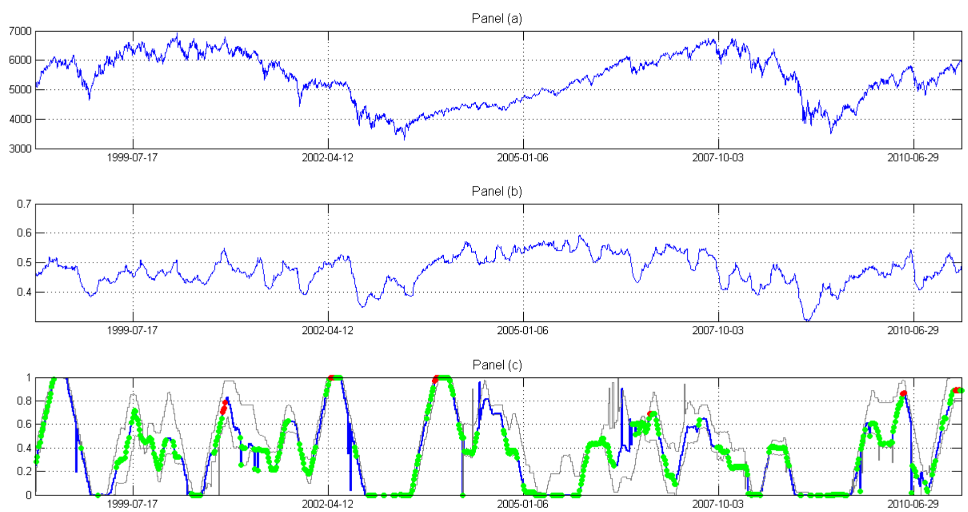

Figure 2: Footsie 100: (a) Price - (b) Estimated $S(t)$ - (c) $\mathscr{E}_{\Delta,\tau}(\alpha)$

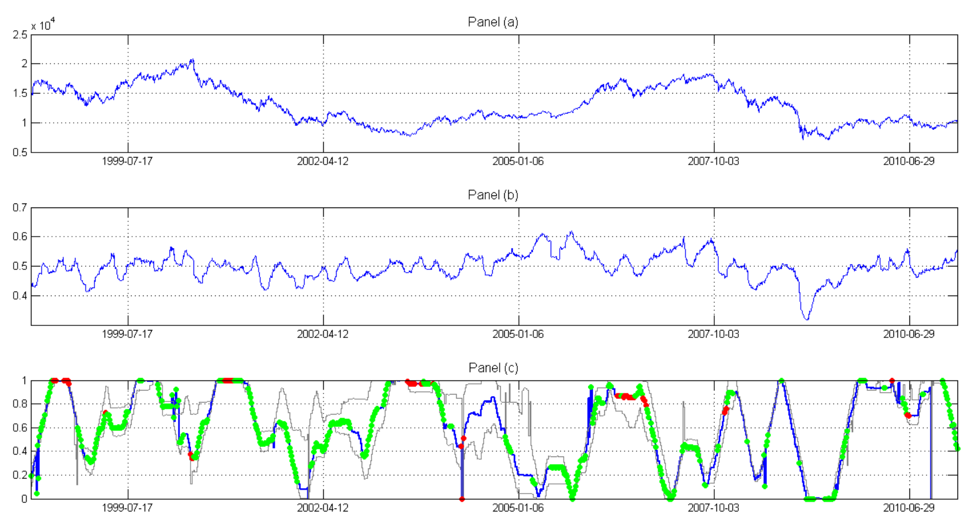

Figure 3: Nikkei 225: (a) Price - (b) Estimated $S(t)$ - (c) $\mathscr{E}_{\Delta,\tau}(\alpha)$