An Adaptive Distributed Protocol for Finite-time Infimum or Supremum Dynamic Consensus

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***Abstract*— In this paper, the problem of distributively tracking the minimum infimum (or maximum supremum) of a set of time-varying signals in finite-time is addressed. More specifically, each agent has access to a local time-varying exogenous signal, and all the agents are required to follow the minimum infimum (or the maximum supremum) of these signals in a distributed fashion. No assumption is made on the network size nor on the bounds of the exogenous signal derivatives. An adaptive protocol is provided which can provably solve the above problem in finite-time for multi-agent systems with undirected connected network topologies. Numerical simulations are provided to corroborate the theoretical findings.**

***Index Terms*— Distributed Control, Adaptive Control, Dynamic Consensus, Multi-Agent Systems**

I. Introduction

Reaching an agreement by resorting only to local interactions is one of the most widely investigated problem in the context of networked multi-agent systems. Over the years, a great attention has been devoted to the consensus problem in all its variations [1]–[10]. Historically speaking, the *static average* consensus formulation, i.e., the problem of requiring the states of all the agents to agree to the average of their initial values, has been the first one to be addressed, e.g. [1], [2]. Successively, the *dynamic average* consensus or *average consensus tracking* formulation, i.e., the problem of tracking the average of time-varying exogenous signals, has been also extensively investigated in the literature, e.g., [3], [4].

Another important class of consensus problems is the *min/max* formulation, for which both a *static* and a *dynamic* version can be identified. In the static version, the objective is to achieve a consensus within the network towards the min (or max) of the agents initial conditions, see for example [5]–[8], while on the dynamic formulation the objective is to track the min (or max) over a set of exogenous signals, each owned by one agent, see for example [9].

In this work, a novel consensus formulation is proposed where the multi-agent system aims to reach an agreement towards the minimum (or maximum) among the infimum (or the supremum) of a set of exogenous signals, each of which is locally available only to one agent of the network. In the following we will refer to these problem formulations as *infimum* and *supremum consensus*, respectively. Notably, as for the previous consensus problems, also for these variations both static and dynamic formulations can be considered.

Interestingly, while the static infimum (or supremum) formulation coincides with the static min (or max) consensus problem, the dynamic infimum (or supremum) formulation differs from the dynamic min (or max) consensus problem since in the proposed formulation the agents are not required to "track up" (or "track down") the minimum (or maximum) signal when it increases (or decreases) as it is required instead for the dynamic min (or max) consensus, respectively.

In this paper, for the reasons mentioned above, we will focus our attention only on the infimum (or supremum) *dynamic* consensus problem. In this regard, to the best of the authors' knowledge, the only relevant reference is [9], where the authors address the dynamic min/max consensus by proposing a discrete-time protocol with convergence error bounds guarantees by assuming exogenous signals with known bounded derivatives. On the contrary, in our setting, each agent is assumed to have an exogenous time-varying signal with *unknown* bounded derivative. In particular, an adaptive protocol is proposed which can solve the dynamic *supremum* or *infimum* consensus problem in finite-time for a multi-agent system with an undirected connected network topology. Numerical simulations are provided in a precision farming setting to corroborate the theoretical findings. In particular, inspired by the needs of the H2020 European project CANOPIES, a scenario where multiple robots needs to monitor upper bounds of signals associated with the execution of tasks is considered. Indeed, this is relevant to ensure timely collaboration to effectively perform agronomic operations.

II. Preliminaries

A. Network Modeling

Let us consider an undirected network of n agents. We model the underlying communication topology via an undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ with set of nodes $\mathcal{V} = \{1, \dots, n\}$ and set of edges $\mathcal{E} = \{(i, j) : i \in \mathcal{V}, j \in \mathcal{V}, i \neq j\}$. Each node $i \in \mathcal{V}$ represents an agent of the network and each edge $(i, j) \in \mathcal{E}$ describes the possibility of communication between pairs of agents. Since $\mathcal{G}$ is undirected, for each edge $(i, j) \in \mathcal{E}$ there exists the edge $(j, i) \in \mathcal{E}$. We denote with

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$\mathcal{N}_i = \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$ the neighborhood of agent i, i.e., the subset of agents which agent i can communicate to, and with $\overline{\mathcal{N}}_i = \mathcal{N}_i \cup \{i\}$ the augmented neighborhood of agent i, comprising the neighbors of the agent and the agent i itself. A graph is called connected if for every pair of nodes $i, j \in \mathcal{V}$ there exists a path connecting them.

Assumption 1. *The undirected graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ encoding the multi-agent network topology is connected at any time.*

B. Nonsmooth Analysis

We review some fundamental notions for nonsmooth analysis. For a comprehensive overview of the topic the reader is referred to [11], [12] and references therein.

Consider the (possibly discontinuous) dynamical system

$$\dot{x}(t) = f(x(t), t), \quad x(0) = x_0, \tag{1}$$

with $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ a measurable and essentially locally bounded function. If the differential equation (1) has discontinuous right-hand side, then, following [13], the solution of (1) is defined in the Filippov sense.

Definition 1 (Filippov Solution)**.** *A vector function $x(\cdot)$ is a solution of* (1) *on a time interval $[t_0, t_i]$ if $x(\cdot)$ is absolutely continuous on $[t_0, t_i]$ and for almost all $t \in [t_0, t_i]$ it holds $\dot{x} \in K[f](x(t), t)$, where the set-valued map $K : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, with $2^{\mathbb{R}^n}$ the set of all subsets of $\mathbb{R}^n$, is defined as*

$$K[f](x(t), t) = \bigcap_{\delta > 0} \bigcap_{\mu\{H\}=0} \overline{\text{co}}\{f(B(x(t), \delta) \setminus H, t)\}, \tag{2}$$

with $\bigcap_{\mu\{H\}=0}$ denoting the intersection over all sets H of Lebesgue measure zero, $B(x(t), \delta)$ the ball of radius δ centered at $x(t)$, and $\overline{\text{co}}$ the convex closure.

We review the notion of Clarke's generalized gradient.

Definition 2 (Clarke's Generalized Gradient [14])**.** *Let $V : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ be a locally Lipschitz continuous function. Its Clarke's generalized gradient at (x, t) is*

$$\partial V(x,t) \triangleq \text{co}\Big\{ \lim_{i \to \infty} \nabla V(x_i, t_i) : (x_i, t_i) \to (x, t), (x_i, t_i) \notin \Omega_V \Big\}, \tag{3}$$

with ∇V the gradient function, $x_i \in \mathbb{R}^n$ a point of an infinite succession converging to x, Ω_V a set of Lebesgue measure zero containing all points where $\nabla V(x)$ does not exist.

Based on the above, we report the chain rule to differentiate Lipschitz regular functions along Filippov's solutions.

Theorem 1 (Chain Rule [11])**.** *Let $x(\cdot)$ be a Filippov solution to* (1) *on an interval containing t and $V : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ be a Lipschitz and regular function. Then, $V(x(t), t)$ is absolutely continuous, $\frac{d}{dt}(V(x(t), t))$ exists almost everywhere and $\frac{d}{dt}(V(x(t), t)) \in^{a.e.} \dot{\tilde{V}}(x(t), t)$ with $\dot{\tilde{V}}(x(t), t)$ defined as*

$$\dot{\tilde{V}}(x(t), t) = \bigcap_{\xi \in \partial V(x(t), t)} \xi^T \begin{pmatrix} K[f](x(t), t) \\ 1 \end{pmatrix}. \tag{4}$$

We additionally report a revised version of the generalized Lyapunov theorem as in [15].

Theorem 2 (Finite-Time Stability Theorem)**.** *Let $x : [t_0, \infty) \to \mathbb{R}^n$ be a Filippov solution to* (1) *and $V : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$, be a time dependent regular function such that $V(x(t), t) = 0$ $\forall x(t) \in \mathcal{C}(t)$ and $V(x(t), t) > 0$ $\forall x(t) \notin \mathcal{C}(t)$, with $\mathcal{C}(t) \subset \mathbb{R}^n$ a compact set. Furthermore, let $x(t)$ and $V(x(t), t)$ be absolutely continuous on $[t_0, \infty)$ with $\frac{d}{dt}(V(x(t), t)) \leq -\varepsilon < 0$ almost everywhere on $\{t : x(t) \notin \mathcal{C}(t)\}$. Then, $V(x(t), t)$ converges to 0 in finite-time and $x(t)$ reaches the compact set $\mathcal{C}(t)$ in finite-time as well.*

Finally, we define the discontinuous sign function of a variable $y \in \mathbb{R}$ and the respective set-valued function SIGN:

$$\text{sign}(y) = \begin{cases} 1 & \text{if } y > 0, \\ 0 & \text{if } y = 0, \\ -1 & \text{if } y < 0, \end{cases} \text{SIGN}(y) \in \begin{cases} 1 & \text{if } y > 0, \\ [-1, 1] & \text{if } y = 0, \\ -1 & \text{if } y < 0. \end{cases} \tag{5}$$

For vector input, we apply the sign functions element-wise. In addition, we use the notation $\text{SIGN}(y) \leq 0$ (≥ 0) to compactly denote that $\phi \leq 0$ (≥ 0), $\forall \phi \in \text{SIGN}(y)$.

C. Problem Setting

Let us introduce the definitions of supremum and infimum.

Definition 3. *Let $g : [t_0, \infty) \to \mathbb{R}$ be a continuous function. The supremum (infimum) of $g(\tau)$ in $[t_0, t]$ is defined as*

$$\overline{g}(t_0, t) := \sup\nolimits_{\tau \in [t_0, t]} g(\tau), \ \big(\underline{g}(t_0, t) := \inf\nolimits_{\tau \in [t_0, t]} g(\tau)\big).$$

From here on, we drop the dependency on the initial time instant t_0, i.e., $\overline{g}(t) = \overline{g}(t_0, t)$ and $\underline{g}(t) = \underline{g}(t_0, t)$.

Let us consider a multi-agent system composed of n interconnected agents, in which each agent i holds a state variable $x_i(t) \in \mathbb{R}$ evolving according to the first-order dynamics $\dot{x}_i(t) = u_i(t)$, where $u_i(t) \in \mathbb{R}$ is the update law. In addition, let us assume that each agent can sense a scalar exogenous signal $r_i(t)$, for which the following assumption holds.

Assumption 2. *The time-varying signals $r_i(t)$ are absolutely continuous, and with locally essentially bounded derivatives. Moreover, there exists an* unknown *constant $\kappa_r \geq 0$ such that for all $t \geq t_0$ and all $\psi \in K[\dot{r}_i](t)$, $\forall i$, it holds $|\psi| < \kappa_r$.*

The above assumption simply states that the absolute value of the derivative ψ of the exogenous signal i, taking values in $K[\dot{r}_i](t)$, $\forall i$, as per Definition 1, is always upper bounded with unknown upperbound.

The distributed finite-time supremum and infimum dynamic consensus problems are defined as follows.

Problem 1. *Consider a multi-agent system with n agents under Assumption 1. We define the finite-time supremum (infimum) dynamic consensus problem as the problem of finding control input $u_i(t)$ such that there exists a finite $\bar{T} \geq 0$ for which the following holds $\forall i \in \mathcal{V}$*

$$|x_i(t) - \overline{r}(t)| = 0, \quad \big(|x_i(t) - \underline{r}(t)| = 0\big), \quad \forall t \geq \bar{T}, \tag{6}$$

where $\overline{r}(t) := \max_{i \in \mathcal{V}} \{\overline{r}_i(t)\}$ $(\underline{r}(t) := \min_{i \in \mathcal{V}} \{\underline{r}_i(t)\})$.

III. Adaptive Distributed Protocol

In the following, we provide a general distributed protocol for solving Problem 1. In order to tackle exogenous signals with *unknown* bounds on the derivatives (see Assumption 2), we define an *adaptive* distributed protocol with time-varying gains. In the case of supremum consensus, we propose that each agent i runs the following update law

$$u_i(t) = -\sum_{j\in I_i^+(t)} \alpha_{ij}(t)\,\mathrm{sign}\,(x_i(t)-x_j(t)) \\ -\beta_i(t)\,\mathrm{sign}\left(x_i(t)-z_i^+(t)\right), \tag{7}$$

with $\alpha_{ij}(t_0), \beta_i(t_0) > 0$, for $j \in \mathcal{N}_i$ and where $I_i^+(t) \subset \mathcal{V}$ is the set of neighbors of agent i with maximum state in the augmented neighborhood, i.e.,

$$I_i^+(t) = \{j \in \mathcal{N}_i \,|\, x_j(t) = \max_{k\in\overline{\mathcal{N}}_i} x_k(t)\}, \tag{8}$$

and $z_i^+(t)$ is the maximum between the agent's state and its local supremum value $\bar{r}_i(t)$, i.e.,

$$z_i^+(t) = \max\left\{x_i(t)\,,\,\bar{r}_i(t)\right\}. \tag{9}$$

The update law in (7) is thus composed of *i)* a consensus term to achieve agreement on the maximum state among the agents, and *ii)* a supremum tracking term to track the maximum supremum reference signal among the agents. Regarding the adaptive gains $\alpha_{ij}(t)$ and $\beta_i(t)$, $\forall i,j \in \mathcal{V}$, the basic idea is to have them grow until local consensus and tracking of the signal $z_i^+(t)$ are achieved, respectively. To this aim, we define the auxiliary variables $h_i^c(t), h_i^z(t) \in \mathbb{R}^+$, $\forall i \in \mathcal{V}$ as

$$h_i^c(t) = \sum_{j\in I_i^+(t)} |x_i(t) - x_j(t)|\,, \\ h_i^z(t) = \left|x_i(t) - z_i^+(t)\right|, \tag{10}$$

and the auxiliary variables $\gamma_{ij}(t)$, $\forall i,j \in \mathcal{V}$, with $\gamma_{ij}(t_0) = 0$ for $(i,j) \notin \mathcal{E}$ and $\gamma_{ij}(t_0) > 0$ for $(i,j) \in \mathcal{E}$, such that

$$\alpha_{ij}(t) = \gamma_{ij}(t) + \gamma_{ji}(t). \tag{11}$$

The following dynamics are selected for $\gamma_{ij}(t)$ and $\beta_i(t)$

$$\dot{\gamma}_{ij}(t) = \begin{cases} \kappa_1 & \text{if } h_i^c(t) > 0 \text{ and } j \in I_i^+(t), \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

$$\dot{\beta}_i(t) = \begin{cases} \kappa_2 & \text{if } h_i^z(t) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ represent the constant growth rates of the adaptive gains, γ_{ij} and β_i, that are applied whenever the functions h_i^c and h_i^z, respectively, are greater than zero. Note that, according to the above definition, it holds $\alpha_{ij}(t) = \alpha_{ji}(t)$ and $\alpha_{ij}(t_0) = 0$ for $(i,j) \notin \mathcal{E}$.

In order for the agents to implement the proposed protocol in (7)-(13), each agent i at time t needs to communicate both the state variable $x_i(t)$ and the auxiliary variable $\gamma_{ij}(t)$ to its neighbors in $j \in \mathcal{N}_i$.

Similarly to the above, in case of infimum consensus, we define the set $I_i^-(t)$ and the variable $z_i^-(t)$ as

$$I_i^-(t) = \{j \in \mathcal{N}_i \,|\, x_j = \min_{k\in\overline{\mathcal{N}}_i} x_k\}, \\ z_i^-(t) = \min\left\{x_i(t)\,,\,\underline{r}_i(t)\right\}. \tag{14}$$

Then, each agent i runs the distributed protocol in (7)-(13) by using $I_i^-(t)$ in place of $I_i^+(t)$ and $z_i^-(t)$ in place of $z_i^+(t)$.

To prove the results, we introduce the stacked vectors $x = [x_1, ..., x_n]^T \in \mathbb{R}^n$, $r = [r_1, ..., r_n]^T \in \mathbb{R}^n$, $\alpha = [\alpha_{11}, \alpha_{21}, ..., \alpha_{nn}]^T \in \mathbb{R}^{n^2}$, $\beta = [\beta_1, ..., \beta_n]^T \in \mathbb{R}^n$ as well as the following sets:

$$I^+(t) = \{j \in \mathcal{V} \,|\, x_j = \max_{i\in\mathcal{V}} x_i(t)\}, \tag{15}$$

$$I^-(t) = \{j \in \mathcal{V} \,|\, x_j = \min_{i\in\mathcal{V}} x_i(t)\}, \tag{16}$$

which correspond to the sets of agents holding the maximum and minimum state values, respectively. In addition, we define the signal $\bar{r}^+(t) = \max_{i\in I^+(t)} \bar{r}_i(t)$. In the following, we omit the time-dependence if not necessary and denote with $\bar{(\cdot)}$ the upperbound of the respective quantity $(\cdot)$. For the sake of space, we only conduct the analysis for the supremum dynamic consensus problem; the analysis for the infimum problem follows a similar reasoning. To demonstrate the result, let us now introduce the following technical result.

Lemma 1. *Consider the* sign *function in* (5)*, and let* $g(y) = \mathrm{sign}(y)$*, with* $y \geq 0$ *(*$y \leq 0$*). Then, it holds:*

$$K[g](y) \in \begin{cases} \{1\} & y > 0 \\ [0,\,1] & y = 0 \end{cases}, \quad \left(K[g](y) \in \begin{cases} \{-1\} & y < 0 \\ [-1,\,0] & y = 0 \end{cases}\right).$$

Proof. The result directly follows by considering that $g(y)$ is such that $g : \mathbb{R} \to [0,\,1]$ ($g : \mathbb{R} \to [-1,\,0]$) and by applying the definition in (2). □

The proof is organized as follows: *i)* we first show that the agents in $I^+(t)$ track $\bar{r}^+(t)$ in finite-time; *ii)* we then prove that a consensus is reached within the multi-agent system in finite-time; and *iii)* we finally prove that by combining these results the proposed protocol solves Problem 1.

Lemma 2. *Consider a multi-agent system with* n *agents. Let Assumptions 1 and 2 hold, and assume* $x_i(t_0) \leq \bar{r}_i(t_0)$*. Consider that each agent runs the protocol in* (7) *with gains evolving according to* (10)-(13)*. Then, for any set of initial conditions,* $\{x(t_0), r(t_0), \beta(t_0)\}$*, there exists a finite time* T' *for which the tracking of the signal* $\bar{r}^+(t)$ *is achieved by the agents in* $I^+(t)$ *with finite upper-bounds* $\bar{\beta}_i$ *on the gains* $\beta_i(t)$*,* $\forall i \in \mathcal{V}$*,* $\forall\, t \geq T'$*.*

Proof. To prove this lemma, let us introduce the following variable $\widetilde{r}(t)$ defined as

$$\widetilde{r}(t) = \bar{r}^+(t) - \frac{1}{|I^+(t)|}\sum_{i\in I^+(t)} x_i(t). \tag{17}$$

We consider the following Lyapunov candidate

$$V_1(x(t),\beta(t),t) = |\widetilde{r}(t)| + \frac{1}{\omega}\sum_{i\in I^+(t)} |\bar{\beta}_i - \beta_i(t)|, \tag{18}$$

with ω a positive constant and $\bar{\beta}_i$ a finite upperbound of β_i, $\forall i \in \mathcal{V}$, to be defined later. To compact the notation, we refer to $V_1(x(t),\beta(t),t)$ simply as V_1 in the following. Note that, by definition, it holds

$$\frac{1}{|I^+|}\sum\nolimits_{i\in I^+(t)} x_i(t) = x_i(t), \quad \forall i \in I^+(t).$$

Moreover, the cardinality of the set $I^+(t)$ is a piecewise constant function having instants of discontinuity that belong to a set of measure zero [16]. This implies that these instants can be disregarded in the nonsmooth analysis and $I^+(t)$ can be studied as a set with constant cardinality. To compute the generalized derivative $\dot{\tilde{V}}_1$ as defined in (4), we need to define its Clarke's generalized gradient ∂V_1 as defined in (3) as well as the set valued maps $K[\dot{x}](x,\beta)$ and $K[\dot{\beta}](x)$. Regarding the generalized gradient, it can be expressed as $\partial V_1 \subseteq \left[\begin{array}{c} \partial_x V_1^T,\, \partial_\beta V_1^T,\, \partial_t V_1 \end{array}\right]^T$, where

$$\begin{aligned}
&\partial_x V_1 \subseteq -\operatorname{SIGN}(\widetilde{r}(t))\frac{1}{|I^+|}s^+,\\
&\partial_\beta V_1 \subseteq -\frac{1}{\omega}\operatorname{diag}(s^+)\operatorname{SIGN}(\bar{\beta}-\beta(t)),\\
&\partial_t V_1 \subseteq \operatorname{SIGN}(\widetilde{r}(t))K[\dot{\bar{r}}^+],
\end{aligned} \tag{19}$$

with $s^+ \in \mathbb{R}^n$ a selection vector with component i equal to 1 if $i \in I^+$, 0 otherwise, $\operatorname{diag}(\cdot)$ a diagonal matrix with values $(\cdot)$ along the main diagonal and $\bar{\beta} = [\bar{\beta}_1, ..., \bar{\beta}_n]^T \in \mathbb{R}^n$. Regarding the set-valued maps, they can be computed as [12]

$$\begin{aligned}
K[\dot{x}](x,\beta) &\subseteq \Big[K[\dot{x}_1](x,\beta),\, K[\dot{x}_2](x,\beta),\, \dots, K[\dot{x}_n](x,\beta)\Big]^T,\\
K[\dot{\beta}](x) &\subseteq \Big[K[\dot{\beta}_1](x,\beta),\, K[\dot{\beta}_2](x,\beta),\, \dots, K[\dot{\beta}_n](x,\beta)\Big]^T,
\end{aligned} \tag{20}$$

where the set-valued map $K[\dot{x}_i](x,\beta)$ is defined as follows

$$\begin{aligned}
K[\dot{x}_i](x,\beta) \subseteq &-\sum_{j\in I_i^+(t)} \alpha_{ij}(t)\operatorname{SIGN}(x_i(t)-x_j(t))\\
&-\beta_i(t)\operatorname{SIGN}(x_i(t)-z_i^+(t)),
\end{aligned} \tag{21}$$

while the set-valued map $K[\dot{\beta}_i](x)$ is defined as

$$K[\dot{\beta}_i](x) = \kappa_2 \operatorname{SIGN}(h_i^z(x(t))). \tag{22}$$

By applying Theorem 1, the generalized derivative $\dot{\tilde{V}}_1$ can be computed as

$$\dot{\tilde{V}}_1\,(x(t),\beta(t),t) = \bigcap_{[\xi_x^T\ \xi_\beta^T\ \xi_t]^T\in\partial V_1} \xi_x^T K[\dot{x}](x) + \xi_\beta^T K[\dot{\beta}](x,\beta) + \xi_t. \tag{23}$$

Let us analyze a generic element of this set-valued derivative which has the following form

$$\begin{aligned}
&\xi_x^T K[\dot{x}](x,\beta) + \xi_\beta^T K[\dot{\beta}](x) + \xi_t \subseteq \operatorname{SIGN}(\widetilde{r})\times\\
&\left(-\frac{1}{|I^+|}\sum_{i\in I^+} K[\dot{x}_i] + K[\dot{\bar{r}}^x]\right) - \frac{1}{\omega}\sum_{i\in I^+}\operatorname{SIGN}(\bar{\beta}_i-\beta_i)K[\dot{\beta}_i],
\end{aligned} \tag{24}$$

where the term $\sum_{i\in I^+} K[\dot{x}_i]$ can be simplified as

$$\textstyle\sum_{i\in I^+} K[\dot{x}_i] \subseteq -\sum_{i\in I^+}\beta_i \operatorname{SIGN}(x_i - z_i^+), \tag{25}$$

due to the symmetric nature of the contributions, i.e., that by construction it holds $\sum_{i\in I^+}\sum_{j\in I_i^+}\alpha_{ij}\operatorname{SIGN}(x_i-x_j)=0$. In view of (22) and (25), the generic element in (24) can be rewritten as

$$\begin{aligned}
&\operatorname{SIGN}(\widetilde{r})\left(\frac{1}{|I^+|}\sum_{i\in I^+}\beta_i\operatorname{SIGN}(x_i-z_i^+) + K[\dot{\bar{r}}^+]\right)\\
&-\frac{1}{\omega}\kappa_2\sum_{i\in I^+}\operatorname{SIGN}(\bar{\beta}_i-\beta_i)\operatorname{SIGN}(h_i^z).
\end{aligned} \tag{26}$$

By virtue of Lemma 1, it holds $\operatorname{SIGN}(h_i^z) \geq 0$, since $h_i^z \geq 0$ in (10), and $\operatorname{SIGN}(x_i - z_i^+) \leq 0$, since $x_i \leq z_i^+$ according to the definition of z_i^+ in (9) for the supremum tracking problem. Let us analyze the case in which $\widetilde{r} > 0$, i.e., the maximum signal $\bar{r}^+$ is not tracked by the agents in I^+. By the definition of $\widetilde{r}$, it holds $x_i < \bar{r}^+$, $\forall i \in I^+$. This implies that there must exist at least one agent $k \in I^+$ for which it holds $\bar{r}_k > x_k$. Therefore, for this agent it holds $h_k^z(t) > 0$ according to the definition in (10), leading to $\beta_k(t) < \bar{\beta}_k$. In view on the above considerations, the set-valued function $\operatorname{SIGN}(\widetilde{r}(t))$ assumes the following value $\operatorname{SIGN}(\widetilde{r}(t)) \subseteq \{1\}$, while for the set-valued functions $\operatorname{SIGN}(\bar{\beta}_i - \beta_i(t))$, $\forall i \in I_i^+$ it holds $\operatorname{SIGN}(\bar{\beta}_k - \beta_k(t)) \subseteq \{1\}$, for $k \in I^+ : \bar{r}_k > x_k$, and $\operatorname{SIGN}(\bar{\beta}_i - \beta_i(t)) \subseteq [0,1]$, for $i \neq k$, $i \in I^+$. To derive an upperbound to the generalized time-derivative $\frac{d}{dt}V_1 \in^{\text{a.e.}} \dot{\tilde{V}}_1$, we assume that the agent k is the one having lowest gain β_i in the network, denoted as β^m. Moreover, in view of Assumption 2, it holds $|\psi| \leq \kappa_r$, $\forall\, \psi \in K[\dot{\bar{r}}^+]$. Based on the above and considering the form in (26), the generalized time-derivative of V_1 can be upper bounded as

$$\dot{V}_1(t) \leq -\frac{1}{|I^+|}\beta^m(t) + \kappa_r - \frac{1}{\omega}\kappa_2 \leq -\frac{1}{n}\beta^m(t) + \kappa_r - \frac{1}{\omega}\kappa_2. \tag{27}$$

At this point, we show that the gains β_i are bounded, $\forall i \in \mathcal{V}$. Let us assume by contradiction that h_i^z for some i in (10) is different than zero for an unlimited time interval; then, it is straightforward to show that β^m would increase with rate κ_2, leading in finite-time to $\dot{V}_1 \leq -\varepsilon'$, with $\varepsilon' > 0$. This shows that V_1 and, then, h_i^z can be different than zero only for a finite-time interval, implying that β_i, $\forall\, i$, are upperbounded by $\bar{\beta}_i$. By selecting $\omega = \kappa_2/\kappa_r$ and by considering that the gains β_i are non-decreasing, $\forall i$, (27) can be rewritten as

$$\dot{V}_1 \leq -\beta^m(t)/n \leq -\beta^m(t_0)/n. \tag{28}$$

This inequality shows that, starting from $x_i(t_0) \leq \bar{r}_i(t_0)$, $\forall i \in \mathcal{V}$, as long as $\widetilde{r}(t) > 0$, the Lyapunov candidate decreases with a constant rate ensuring that the condition $\widetilde{r}(t) = 0$ is reached in a finite-time T', for which it holds

$$V_1(t) \leq V_1(t_0) - \int_{t_0}^{t}\frac{\beta^m(t_0)}{n}d\tau, \Rightarrow\, T' \leq \frac{n\,V_1(t_0)}{\beta^m(t_0)} + t_0. \tag{29}$$

□

Lemma 3. *Consider a multi-agent system with n agents under Assumption 1. Consider that each agent runs the protocol in* (7) *with gains evolving as in* (10)-(13)*. Then, for any set of initial conditions, $\{x(t_0), \alpha(t_0)\}$, there exists a finite-time T for which consensus is reached with finite upper-bounds $\bar{\alpha}_{ij}$ on the gains $\alpha_{ij}(t)$, $\forall (i,j) \in \mathcal{E}$, $\forall\, t \geq T$.*

Proof. To prove this lemma, we follow a similar reasoning as in [16], [17] which is extended to the case of adaptive gains. Let us consider the following Lyapunov candidate

$$\begin{aligned}
V_2(x(t),\alpha(t)) = \frac{1}{|I^+(t)|}&\sum_{i\in I^+(t)} x_i(t) - \frac{1}{|I^-(t)|}\sum_{i\in I^-(t)} x_i(t)\\
&+\frac{1}{\sigma}\sum_{i\in I^-(t)}|\bar{\alpha}_{ij} - \alpha_{ij}(t)|,
\end{aligned} \tag{30}$$

with σ a positive constant. As in the previous lemma, we can disregard the instants where the cardinalities $I^+(t)$ and $I^-(t)$ are discontinuous in the nonsmooth analysis. We denote $V_2(x(t), \alpha(t))$ simply as V_2 in the following. The Clarke's generalized gradient ∂V_2 given in (3) can be expressed as $\partial V_2 \subseteq \left[\partial_x V_2^T, \partial_\alpha V_2^T\right]^T$ with

$$\begin{aligned} \partial_x V_2 &= \frac{1}{|I^+|} s^+ - \frac{1}{|I^-|} s^-, \\ \partial_\alpha V_2 &\subseteq -\frac{1}{\sigma}\left[\operatorname{diag}\left(s^-\right) \otimes I_n\right] \operatorname{SIGN}(\bar{\alpha} - \alpha). \end{aligned} \tag{31}$$

where s^+ is defined as in (19), $s^- \in \mathbb{R}^n$ is a selection vector with i th component equal to 1 if $i \in I^-$, 0 otherwise, and $\bar{\alpha}$ is the stacked vector of the gains upperbounds $\bar{\alpha}_{ij}$, $\forall i, j \in \mathcal{V}$. The set-valued map $K[\dot{\alpha}](x)$ can be computed as [12]

$$K[\dot{\alpha}](x) \subseteq \Big[K[\dot{\alpha}_{11}](x),\, K[\dot{\alpha}_{21}](x),\, \ldots, K[\dot{\alpha}_{nn}](x)\Big]^T,$$

with $K[\dot{\alpha}_{ij}](x)$ defined as [12]

$$K[\dot{\alpha}_{ij}](x) \subseteq K[\dot{\gamma}_{ij}](x) + K[\dot{\gamma}_{ji}](x), \tag{32}$$

with

$$K[\dot{\gamma}_{ij}](x) = \begin{cases} \kappa_1 \operatorname{SIGN}(h_i^c(x)) & \text{if } j \in I_i^+, \\ 0 & \text{if } j \notin I_i^+. \end{cases} \tag{33}$$

Therefore, the generalized derivative $\dot{\tilde{V}}_2$ can be computed according to Theorem 1 as

$$\dot{\tilde{V}}_2\left(x(t), \alpha(t)\right) = \bigcap_{[\xi_x^T\ \xi_\alpha^T]^T \in \partial V_2} \xi_x^T K[\dot{x}](x, \alpha) + \xi_\alpha^T K[\dot{\alpha}](x).$$

In view of the form of the generalized gradient in (31), we can analyze the term $\xi_x^T K[\dot{x}](x, \alpha) + \xi_\alpha^T K[\dot{\alpha}](x)$ as follows

$$\begin{aligned} \xi_x^T K[\dot{x}](x, \alpha) + \xi_\alpha^T K[\dot{\alpha}](x) \subseteq \underbrace{\frac{1}{|I^+|} \sum_{i \in I^+} K[\dot{x}_i]}_{\dot{V}_x^+} \\ - \underbrace{\frac{1}{|I^-|} \sum_{i \in I^-} K[\dot{x}_i]}_{\dot{V}_x^-} - \underbrace{\frac{1}{\sigma} \sum_{i \in I^-} \sum_{j \in I_i^+} \operatorname{SIGN}(\bar{\alpha}_{ij} - \alpha_{ij}) K[\dot{\alpha}_{ij}]}_{\dot{V}_\alpha}. \end{aligned}$$

In light of (21) and the symmetry condition, the term $\dot{V}_x^+$ can be rewritten as

$$\dot{V}_x^+ = -\frac{1}{|I^+|} \sum_{i \in I^+} \beta_i \operatorname{SIGN}(x_i - z_i^+). \tag{34}$$

Regarding the term $\dot{V}_x^-$, this can be formulated as

$$\dot{V}_x^- = -\tfrac{1}{|I^-|} \sum_{i \in I^-} \left(\sum_{j \in I_i^+} \alpha_{ij} \operatorname{SIGN}(x_i - x_j) + \beta_i \operatorname{SIGN}(x_i - z_i^+) \right). \tag{35}$$

For the term $\dot{V}_\alpha$, we obtain the following from (32)

$$\dot{V}_\alpha \leq -\frac{1}{\sigma} \sum_{i \in I^-} \sum_{j \in I_i^+} \kappa_1 \operatorname{SIGN}(h_i^c), \tag{36}$$

where we exploited the fact that $\phi \leq \psi$ with $\phi \in K[\dot{\alpha}_{ij}]$, $\psi \in K[\dot{\gamma}_{ij}]$ according to the definition in (11). At this point, we want to show that as long as $I^+ \cap I^- = \emptyset$, i.e., agents are not at consensus, the Lyapunov candidate is decreasing with a constant rate. To this aim, we analyze the worst case scenario for the function to decrease. Regarding the terms related to the set I^+, recalling that $\operatorname{SIGN}(x_i - z_i^+) \leq 0$, we observe that in the worst case for all the agents in I^+ it holds $x_i < z_i^+$. Regarding the terms related to the set I^-, the condition $I^+ \cap I^- = \emptyset$ implies that there exists at least one agent $i \in I^-$ with a neighbor $j \notin I^-$ with $x_j > x_i$, for which it holds $h_i^c > 0$ and $\alpha_{ij} < \bar{\alpha}_{ij}$, i.e., the set $\mathcal{S}(t) = \{i \in I^-(t) \,|\, \mathcal{N}_i \cap (\mathcal{V} \setminus I^-(t)) \neq \emptyset\}$ is non-empty. Let $\alpha^{\mathcal{S}_m}(t)$ be the minimum gain $\alpha_{ij}(t)$ such that $i \in \mathcal{S}(t)$, $\forall j \in \mathcal{N}_i$. Based on these considerations and on the expressions in (34), (35) and (36), we obtain that as long as $I^+ \cap I^- = \emptyset$, the following inequalities hold true

$$\dot{V}_2(t) \leq \beta^M - \frac{\alpha^{\mathcal{S}_m}(t)}{|I^-|} - \frac{1}{\sigma}\kappa_1 \leq \bar{\beta}^M - \frac{\alpha^m(t_0)}{n} - \frac{1}{\sigma}\kappa_1, \tag{37}$$

where the subscripts M and m denote the maximum and minimum value of the respective quantity in the network, respectively, and $\alpha_{ij}(t) \geq \alpha^m(t_0)$ follows by construction, $\forall (i,j) \in \mathcal{E}, \forall t \geq t_0$. Similarly to the previous lemma, we now show that the gains α_{ij} are bounded, $\forall (i,j) \in \mathcal{E}$. Let us assume by contradiction that h_i^c in (10) for some i is different than zero for an unlimited time interval; therefore, $\alpha^{\mathcal{S}_m}$ is piece-wise increasing with rate κ_1, leading in finite-time to $\dot{V}_2 \leq -\varepsilon$, with $\varepsilon > 0$. This shows that V_2 and, then, h_i^c can be different than zero only for a finite-time interval, implying that α_{ij}, $\forall i, j$, are upperbounded by $\bar{\alpha}_{ij}$. Finally, let us set $\sigma = \kappa_1 / \bar{\beta}^M$. Then, (37) can be rewritten as

$$\dot{V}_2 \leq -\alpha^m(t_0)/n, \tag{38}$$

which shows that V_2 converges to zero in finite-time T, that is

$$V_2(t) \leq V_2(t_0) - \int_{t_0}^{t} \frac{\alpha^m(t_0)}{n} d\tau, \Rightarrow\ T \leq \frac{n\, V_2(t_0)}{\alpha^m(t_0)} + t_0. \tag{39}$$

We can now state our main result. □

Theorem 3. *Consider a multi-agent system composed of n agents running the protocol in* (7) *with gains evolving according to* (10)-(13). *Assume that the conditions of Lemmas 2 and 3 hold. Then, all the agents track the maximum supremum $\bar{r}$ in finite-time $\bar{T} = \max(T', T)$, with T' and T upperbounded as in* (29) *and* (39)*, respectively.*

Proof. The proof can be derived by combining Lemmas 2 and 3. More specifically, Lemma 3 states that all the agents reach consensus, leading in finite-time to $I^+ \equiv \mathcal{V}$. However, by virtue of Lemma 2, the maximum signal $\bar{r}^+$ is tracked by the maximum agents in I^+ in finite-time, implying that the maximum supremum signal $\bar{r}$ will be tracked by all the agents. Finally, the convergence time is given by the slowest between T' and T in Lemmas 2 and 3, respectively. □

IV. Simulation results

As mentioned in Section I, for the numerical validation of the proposed protocol, motivated by the needs of the H2020

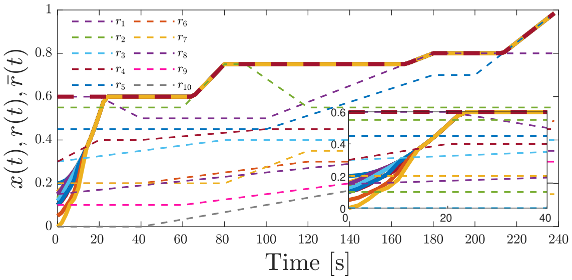

Fig. 1: Evolution of states $x(t)$ (solid lines), filling signals $r_i(t)$ (fine dotted lines) and signal $\bar{r}(t)$ (thick dotted red line).

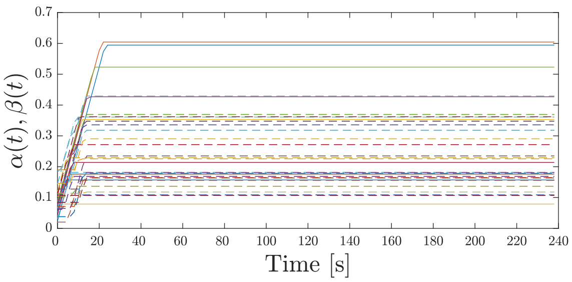

Fig. 2: Evolution of the adaptive gains $\alpha(t)$ (dotted lines) and $\beta(t)$ (solid lines).

European project CANOPIES, we consider a heterogeneous multi-robot team for a precision farming setting in which robots need to monitor the maximum supremum of signals to plan their intervention. The development of distributed protocols for this kind of setting is crucial to achieve scalability over large-scale fields. In particular, according to CANOPIES project view, we consider two kinds of robotic platforms: farming and logistic. Farming robots carry out agronomic operations, like harvesting, filling on-board boxes with collected items. Logistic robots are responsible to empty these boxes to enable the first robots to proceed with farming operations. We consider a system composed of $n = 10$ agents (with connected undirected communication graph) where each agent has access to a filling signal $r_i(t)$ modeling the percentage of filled box (with maximum value 1). The boxes can be partially or totally emptied at any time, i.e., the filling signals can decrease over time. Our distributed algorithm allows to track the maximum supremum $\bar{r}(t)$ of these filling signals, which can activate in a timely manner the intervention of logistic robots to empty the boxes. The filling signals r_i are modelled as piecewise linear functions, depicted as fine dotted lines in Figure 1. We consider the following initial states $x(0) = [0.2 \;\; 0.1 \;\; 0.16 \;\; 0.15 \;\; 0.13 \;\; 0.12 \;\; 0.1 \;\; 0.1 \;\; 0.05 \;\; 0]^T$, while the adaptive gains $\gamma_{ij}(0)$ and $\beta_i(0)$ are initialized with uniform distribution in the interval $[0, 1]$, for all $\forall i \in \mathcal{V}, j \in \mathcal{N}_i$. Also, we set $\kappa_1 = \kappa_2 = 0.5$ in (12) and (13). Figure 1 depicts the state's trajectories (solid lines), the filling signals $r_i(t), \forall i \in \mathcal{V}$ (fine dotted lines) and the maximum supremum signal $\bar{r}$ (thick red dotted line). The figure shows the network reaching consensus in $T \approx 12$ s and tracking the maximum supremum in $T' \approx 22$ s. Moreover, the figure shows that the network keeps tracking the maximum supremum signal in response to variations of the maximum supremum value. An example of this behaviour arises at approximately 60 s when the filling signal r_2 (dotted green line) increases, leading to transition from $\bar{r} = r_1$ (dotted purple line) to $\bar{r} = r_2$. Furthermore, the agents are able to track the maximum supremum signal also when it is greater than all the filling signals. Such case occurs, for instance, in the time intervals $[20, 70]$ s and $[90, 170]$ s. Finally, Figure 2 shows the adaptive gains $\alpha(t)$ (dotted lines) and $\beta(t)$ (solid lines). We can observe that the gains keep increasing as long as the conditions h_i^c and h_i^z in (10) are greater than 0, $\forall i \in \mathcal{V}$, and then reach constant values at about $t \approx 22$ s, allowing to track the maximum supremum signal.

V. Conclusions

In this paper, the finite-time infimum (supremum) dynamic consensus has been proposed. A system composed of n agents interacting over a connected undirected network topology is considered. Each agent has access to a local time-varying exogenous signal with unknown bounded derivative. An adaptive finite-time distributed protocol has been designed. Numerical simulations on a precision farming setting have been considered to corroborate the theoretical findings. Future work will focus on considering directed communication graphs and on ensuring robustness to perturbations or attacks.

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