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# Level and pseudo-Gorenstein path polyominoes

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**Abstract** We classify path polyominoes which are level and pseudo-Gorenstein. Moreover, we compute all level and pseudo-Gorenstein simple thin polyominoes with rank less than or equal to 10. We also compute the regularity of the pseudo-Gorenstein simple thin polyominoes in relation to their rank.

**Keywords** Simple thin polyomino, path polyomino, Cohen-Macaulay, Level, pseudo-Gorenstein

AMS Subject Classification (2020): 13D02, 13C14, 05E40

## 1 Introduction

The ideals generated by a subset of 2-minors of an $m \times n$ matrix of indeterminates are an intensively-studied class of binomial ideals, due to their applications in algebraic statistics. Among these ideals, one finds the determinantal ideals, see, for instance, [2] and its references to original articles, the ladder ideals introduced by Conca in [4], and the ideals of adjacent minors introduced by Hoşten and Sullivan in [13]. More recently two new classes of ideals of this kind were introduced: the binomial edge ideals by Herzog et al. in [9], and independently by Ohtani in [16] and polyomino ideals by Qureshi

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in [17]. A nice survey on these ideals is the book [10]. We recall that polyominoes arise from two-dimensional objects obtained by joining edge by edge unitary squares (see [7]). Over the last few years, algebraic properties of polyomino ideals have been investigated. Within them, one of the nicest result is that simple polyominoes, roughly speaking polyominoes without holes, are Cohen-Macaulay domains (see [11,19]).

Moreover, under the assumption that the polyomino $\mathcal{P}$ is simple and thin, namely it does not contain a square tetromino as subpolyomino, the first and second authors, described the Hilbert-Poincaré series of $S/I_{\mathcal{P}}$ (see Section 2 for the definition of the ring) in terms of the rook complex defined on $\mathcal{P}$ (see [20]). Thanks to this observation, they characterized the Gorenstein simple thin polyominoes. Until now only some other cases of Gorenstein polyominoes are known (see [1,3,6,17,18]).

There are two interesting generalizations of Gorenstein rings: level rings (see [24]) and pseudo-Gorenstein rings (see [5]). Observing that a ring is Gorenstein if and only if the canonical module is a cyclic module, and hence generated in a single degree, the two generalizations naturally arise. In fact, if one only requires that the generators of the canonical module are of the same degree, then the ring is called level, and if one requires that there is only one generator of least degree, then we call it pseudo-Gorenstein. A first study on this topic on binomial edge ideals has been carried on by the first and third authors (see [22]).

As stated by Herzog and others (see [5]), since pseudo-Gorenstein rings can be identified by the property that the leading coefficient of the numerator polynomial of the Hilbert series is equal to 1, pseudo-Gorenstein rings are much easier accessible than level rings. This assertion is in particular true for simple thin polyominoes. In fact they can be described by the existence of a unique configuration of non-attacking rooks of maximum cardinality (see Lemma 1).

Most of our paper is dedicated to the classification of level (see Theorem 10) and pseudo-Gorenstein (see Theorem 4) simple polyominoes that are paths.

After giving the necessary preliminaries (see Section 2), in Section 3 we give a complete description of pseudo-Gorenstein path polyominoes in terms of the non-existence of odd stair, where a stair is a sequence of intervals of length 2 inside the path.

To reach the next goal, that is the classification of path polyominoes that are level, in Section 4 we prove that the socle of the ring $S/I_{\mathcal{P}}$ modulo some linear forms is generated in the same degree. We prove that the non-attacking rooks defined on $\mathcal{P}$ play a fundamental role on the description of the pathological behavior of the stairs. In particular, we prove that any path polyomino without stair is level. By some technical lemmas we describe all pathological stairs, namely the bad stairs, whose existence in the path polyomino make it non-level. These are the ones having length 4,6, or length greater than or equal to 8. Thanks to bad stair we obtain the classification of path polyominoes that are level (see Theorem 10).

In the last section, we focus on simple thin polyominoes. We share the computation that has been done regarding the classification of Gorenstein, level and pseudo-Gorenstein simple thin polyominoes. The result of the computation is downloadable from [21]. Inspired by this computation and from the results on path polyominoes, we describe the regularity of the pseudo-Gorenstein simple thin polyominoes in relation to their rank (see Theorem 11), and we also present a conjecture on level simple thin polyominoes.

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## 2 Polyominoes and Rook complex

Let $a=(i,j), b=(k,\ell)\in\mathbb{N}^2$, with $i\leq k$ and $j\leq \ell$, the set $[a,b]=\{(r,s)\in\mathbb{N}^2 : i\leq r\leq k \text{ and } j\leq s\leq \ell\}$ is called an *interval* of $\mathbb{N}^2$. If $i<k$ and $j<\ell$, $[a,b]$ is called a *proper interval*, and the elements $a,b,c,d$ are called corners of $[a,b]$, where $c=(i,\ell)$ and $d=(k,j)$. In particular, $a,b$ are called *diagonal corners* and $c,d$ *anti-diagonal corners* of $[a,b]$. The corner $a$ (resp. $c$) is also called the left lower (resp. upper) corner of $[a,b]$, and $d$ (resp. $b$) is the right lower (resp. upper) corner of $[a,b]$. A proper interval of the form $C=[a,a+(1,1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a+(1,0), a+(0,1), a+(1,1)$. The sets $\{a,a+(1,0)\}, \{a,a+(0,1)\}, \{a+(1,0),a+(1,1)\}$, and $\{a+(0,1),a+(1,1)\}$ are called the *edges* of C. Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let $C$ and $D$ be two cells of $\mathcal{P}$. Then $C$ and $D$ are said to be *connected* if there is a sequence of cells $C=C_1,\dots,C_m=D$ of $\mathcal{P}$ such that $C_i\cap C_{i+1}$ is an edge of $C_i$ for $i=1,\dots,m-1$. A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P})=\cup_{C\in\mathcal{P}}V(C)$ the vertex set of $\mathcal{P}$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and it is denoted by $\mathrm{rk}(\mathcal{P})$.

A polyomino $\mathcal{Q}$ is called a *subpolyomino* of a polyomino $\mathcal{P}$ if each cell belonging to $\mathcal{Q}$ also belongs to $\mathcal{P}$, and we write $\mathcal{Q}\subseteq\mathcal{P}$. A proper interval $[a,b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a,b]$ belong to $\mathcal{P}$. We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells $C$ and $D$ of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a sequence of cells $C=C_1,\dots,C_m=D$ such that $C_i\notin\mathcal{P}$ for any $i=1,\dots,m$.

We say that two vertices $a,b\in V(\mathcal{P})$ are *diagonally opposite*, or simply *opposite*, if they are either diagonal or antidiagonal corners of an inner interval of $\mathcal{P}$.

A maximal inner interval $[a,b]$ of $\mathcal{P}$ with $a=(i,j)$, $b=(k,\ell)$ and either $k-i=1$ or $\ell-j=1$ is identified as a row or column of cells, called *cell interval*. Let $\mathcal{P}$ be a simple polyomino. We say that a cell $C$ of $\mathcal{P}$ is a *leaf* if there exists an edge $\{u,v\}$ of $C$ such that $\{u,v\}\cap V(\mathcal{P}\setminus\{C\})=\varnothing$. We call the vertices $u$ and $v$ *leaf corners* of $C$.

We say that a polyomino $\mathcal{P}$ is *thin* (see [15], [20]) if $\mathcal{P}$ does not contain the square tetromino (see Figure 1) as a subpolyomino.

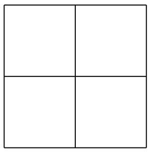


Fig. 1: The square tetromino

In a thin polyomino, any maximal interval is a cell interval. For $k \in \mathbb{N}$, $k$ rooks on the cells of a polyomino $\mathcal{P}$ are said to be *non-attacking* if they do not lie on the same row or column of cells of $\mathcal{P}$, pairwise. The maximum number of non-attacking rooks is called the *rook number* of $\mathcal{P}$, denoted by $r(\mathcal{P})$. We identify the rooks that can be placed on $\mathcal{P}$ with the cells of $\mathcal{P}$. The set $\mathcal{R}_{\mathcal{P}}$ of sets of non-attacking rooks is a simplicial complex and it is called *rook complex.* Let $r_k$ be the number of configurations of $k$-non attacking rooks. The polynomial

$$r_{\mathcal{P}}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k$$

is called the *rook polynomial* of $\mathcal{P}$. We denote by $\mathcal{F}(\mathcal{R}_{\mathcal{P}})$ the set of facets of the rook complex $\mathcal{R}_{\mathcal{P}}$.

We recall the following notation from [23]. Let $\mathcal{C} = \{I_1, \dots, I_s\}$ be the set of the cell intervals of $\mathcal{P}$.

**Definition 1** Let $\mathcal{P}$ be a polyomino. A subset $\mathcal{A} \subseteq \mathcal{C}$ is called a *partition* of $\mathcal{P}$ if

1. $\forall I, J \in \mathcal{A}$ we have $I \cap J = \varnothing$;
2. $\bigcup_{I \in \mathcal{A}} I = \mathcal{P}$.

**Definition 2** An interval $I = \{C_1, \dots, C_m\} \in \mathcal{C}$ is called *embedded* if there exists $F = \{D_1, \dots, D_m\} \in \mathcal{R}_{\mathcal{P}}$ such that for any $i \in \{1, \dots, m\}$ the set $\{C_i, D_i\}$ is attacking.

*Remark 1* Let $I \in \mathcal{C}$ be a non-embedded interval. Then any facet $F \in \mathcal{R}_{\mathcal{P}}$ is such that $F \cap I \neq \varnothing$.

**Definition 3** Let $\mathcal{A}$ be a partition of $\mathcal{P}$. If no interval of $\mathcal{A}$ is embedded then $\mathcal{A}$ is called *super partition.* Moreover we call $\mathcal{P}$ *superpartitionable.*

With the help of superpartitions one can characterize the polyominoes having a pure rook complex $\mathcal{R}_{\mathcal{P}}$.

**Theorem 1 ([23], Theorem 3.10)** *Let $\mathcal{P}$ be a polyomino. The following are equivalent:*

*(1)* $\mathcal{R}_{\mathcal{P}}$ *is pure and has dimension* $r-1$*;*
*(2)* $\mathcal{P}$ *admits a super partition* $\mathcal{A}$ *with* $|\mathcal{A}| = r$*.*

Let us recall some definitions from commutative algebra which will be used through out the paper. Let $M$ be a finitely generated graded $S$-module, where $S$ is a polynomial ring over an arbitrary field $\mathbb{K}$. Let $\mathfrak{m}$ be the homogeneous maximal ideal in $S$. Then the *socle* of $M$ is defined as $\mathrm{Soc}(M) := (0 :_M \mathfrak{m}) = \{z \in M \mid \mathfrak{m}z = 0\}$.

Two important homological invariants, projective dimension and Castelnuovo-Mumford regularity can be computed directly from the Betti table. Let $\beta_{i,i+j}(M)$ be the graded Betti numbers of $M$. Then the *projective dimension* of $M$ is defined as $\mathrm{pdim}(M) := \max\{i : \beta_{i,i+j}(M) \neq 0 \text{ for some } j\}$ and the *Castelnuovo-Mumford regularity* (or simply, *regularity*) of $M$ is defined as $\mathrm{reg}(M) := \max\{j : \beta_{i,i+j}(M) \neq 0 \text{ for some } i\}$. We now recall the definition of a level ring which is intermediate between Cohen-Macaulay and Gorenstein.

**Definition 4** Let $T$ be a graded Cohen-Macaulay $\mathbb{K}$-algebra with $p = \mathrm{pdim}(T)$ and $r = \mathrm{reg}(T)$. Then $T$ is called level if $\beta_{p,p+i}(T) = 0$ for $i < r$.

Let $\mathcal{P}$ be a polyomino. Let $\mathbb{K}$ be an arbitrary field and $S = \mathbb{K}[x_v : v \in V(\mathcal{P})]$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where $c, d$ are the anti-diagonal corners of $[a, b]$. We denote by $\mathcal{M}$ the set of all inner 2-minors of $\mathcal{P}$. The ideal generated by $\mathcal{M}$ in $S$ is said to be the *polyomino ideal* of $\mathcal{P}$ and it is denoted by $I_{\mathcal{P}}$. The properties of $I_{\mathcal{P}}$ and $S/I_{\mathcal{P}}$ arise from combinatorial properties of $\mathcal{P}$.

Let us consider simple thin polyominoes. In [20], the authors study simple thin polyominoes and prove the following:

**Theorem 2 (Theorem 1.1)** *Let* $\mathcal{P}$ *be a simple thin polyomino such that the reduced Hilbert-Poincaré series of* $S/I_{\mathcal{P}}$ *is*

$$\mathrm{HP}_{S/I_{\mathcal{P}}}(t) = \frac{h(t)}{(1-t)^d}.$$

*Then* $h(t)$ *is the rook polynomial of* $\mathcal{P}$*.*

In the same article, the authors introduce a property that is fundamental to characterize Gorenstein simple thin polyominoes.

**Definition 5** Let $\mathcal{P}$ be a simple thin polyomino. A cell $C$ of $\mathcal{P}$ is *single* if there exists a unique maximal inner interval of $\mathcal{P}$ containing $C$. If any maximal inner interval of $\mathcal{P}$ has exactly one single cell, we say that $\mathcal{P}$ has the *S-property.*

**Theorem 3** *Let* $\mathcal{P}$ *be a simple thin polyomino. Then the following conditions are equivalent:*

*(1)* $S/I_{\mathcal{P}}$ *is Gorenstein;*
*(2)* $\mathcal{P}$ *has the* $S$*-property.*

We now give the definition of the main object of the paper.

**Definition 6** A simple polyomino $\mathcal{P}$ is called a path if $\mathcal{P} = \{C_1, \dots, C_\ell\}$ such that

1. $C_i \cap C_{i+1}$ is a common edge for all $i = 1, \dots, \ell$;
2. $C_i \neq C_j$ for all $i \neq j$;
3. For all $i \in \{3, \dots, \ell - 2\}$ and $j \notin \{i-2, i-1, i, i+1, i+2\}$, one has $C_i \cap C_j = \emptyset$.

We denote a path polyomino by $\mathcal{P} = C_1 C_2 \cdots C_\ell$. If $I_1, \dots, I_s$ are the maximal intervals of $\mathcal{P}$, then we denote by $l_k$ the *length* of $I_k$, say $l_k = |I_k|$ for $k \in \{1, \dots, s\}$.

An example of path polyomino is shown in Figure 2.

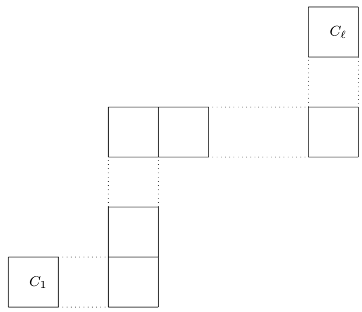



Fig. 2: A path polyomino $\mathcal{P}$

## 3 Pseudo-Gorenstein path polyominoes

We start this section by the following notion that is fundamental for the characterization of pseudo-Gorenstein, and level, path polyominoes.

**Definition 7** Let $\mathcal{S}$ be a path polyomino as in Definition 6 and let $\mathcal{C} = \{I_1, \dots, I_\lambda\}$ be the set of its cell intervals. Then $\mathcal{S}$ is called a *stair*, if $\lambda \geq 3$ and $l_i = 2$ for all $1 < i < \lambda$. The length of the stair is $\lambda$ and $\mathcal{S}$ has odd (resp. even) length if $\lambda$ is odd (resp. even). We denote by $\mathcal{S}_\lambda$ the stair with $l_1 = 2 = l_\lambda$ and by $\tilde{\mathcal{S}}_\lambda$ a stair with $l_1 > 2$ or $l_\lambda > 2$.

A path polyomino $\mathcal{P}$ *has a stair* $\mathcal{S}$, if $\mathcal{S}$ is a stair subpolyomino of $\mathcal{P}$, and $\mathcal{S}$ is not a subpolyomino of any stair of greater length contained in $\mathcal{P}$. Given a path polyomino $\mathcal{P}$, we say that $\mathcal{P}$ *has no odd stairs* if $\mathcal{P}$ has no stairs of odd length.

For some examples of stairs see Figure 3 and Figure 8. The following is a consequence of [20].

**Lemma 1** *Let $\mathcal{P}$ be a simple thin polyomino. Then $S/I_{\mathcal{P}}$ is pseudo-Gorenstein if and only if there exists a unique configuration of non-attacking rooks of maximum cardinality.*

*Remark 2* Let $\mathcal{P}$ be a simple thin polyomino such that $S/I_{\mathcal{P}}$ is pseudo-Gorenstein. Then any interval has at most one single cell. Moreover, let $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ be the unique facet of maximum cardinality of the pseudo-Gorenstein ring $S/I_{\mathcal{P}}$. Then $F$ contains all single cells of $\mathcal{P}$. Indeed, suppose $D_i$ is a single cell contained in the interval $I_i$ and let $C_i \in F$ with $C_i \in I_i$. Then

$$(F \setminus \{C_i\}) \cup \{D_i\}$$

is a facet. That is $C_i = D_i$.

**Theorem 4** *Let $\mathcal{P}$ be a path polyomino with $\mathcal{C} = \{I_1, I_2, \dots, I_s\}$. Then $S/I_{\mathcal{P}}$ is pseudo-Gorenstein if and only if either $\mathcal{P}$ is a cell or the following conditions hold:*

1. *$l_1 = l_s = 2$ and $l_k \leq 3$ for all $2 \leq k \leq s-1$;*
2. *$\mathcal{P}$ does not have odd stairs.*

*Proof* The case $\mathcal{P}$ is a cell is obvious being a principal ideal. Hence from now on we assume $\mathcal{P}$ is not a cell. We observe that if $s = 1$ and $\operatorname{rk}\mathcal{P} > 1$, then $\mathcal{P}$ is not pseudo-Gorenstein.

$\Rightarrow$) Since $\mathcal{P}$ is not a cell, then $s > 1$. Moreover, by Remark 2, every interval has at most one single cell, and since $\mathcal{P}$ is a path, then $l_1 = l_s = 2$ and $l_k \leq 3$. That is (1) holds. Now, assume that $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ is the unique facet of maximal cardinality. Let $I_{h+1}, I_{h+2}, \dots, I_{h+\ell}$ be intervals of an odd stair, with first interval $I_{h+1}$, last interval $I_{h+\ell}$ and $\ell$ odd. Then we may assume that $\{C_h, C_{h+1}\} \subseteq I_{h+1}$, $\{C_{h+2}, C_{h+3}\} = I_{h+3}$, $\{C_{h+4}, C_{h+5}\} = I_{h+5}, \dots$, $\{C_{h+\ell-1}, C_{h+\ell}\} \subseteq I_{h+\ell}$, and $\mathcal{P}' = C_h C_{h+1} \cdots C_{h+\ell}$ is a subpath of $\mathcal{P}$. By the definition of stair, the cells $C_h, C_{h+\ell}$ are single and, since $F$ is unique, then by Remark 2 it follows that $C_h, C_{h+\ell}$ are in $F$. Now,

$$G = (F \setminus \mathcal{P}') \cup \{C_h, C_{h+2}, \dots, C_{h+\ell-1}\}$$

has the same cardinality of $F$ and it is a facet, too. Hence $F$ is not unique. That is (2) holds, too.

$\Leftarrow$) By the Lemma 1, we need to prove that there exists a unique configuration of non-attacking rooks of maximum cardinality. We claim that if (1) and (2) are satisfied then $\mathcal{P} = C_1 \cdots C_{2m+1}$ and the unique maximal facet is $F = \{C_1, C_3, \dots, C_{2m+1}\}$. Suppose there are not even stairs, namely $\mathcal{P}$ does not contain any stair. Then $\mathcal{P}$ is Gorenstein and $F$ has only single cells: $C_1$ of the interval $I_1 = \{C_1, C_2\}$, $C_3$ of $I_2 = \{C_2, C_3, C_4\}$, and so on, ending with $C_{2m+1}$ in $I_s = \{C_{2m}, C_{2m+1}\}$. Now, suppose that $\mathcal{P}$ has at least an even stair and without loss of generality that the first even stair is in the first $\ell$ intervals that is $I_1 = \{C_1, C_2\}$, $I_2 = \{C_2, C_3\}$, and so on until the last one that is $I_\ell = \{C_\ell, C_{\ell+1}, C_{\ell+2}\}$ with $\ell$ even. Since $F$ is maximal, then $C_{\ell+1}$ must belong to $F$. Moreover, $F$ having maximum cardinality contains $\{C_1, C_3, \dots, C_{\ell-1}, C_{\ell+1}\}$. Using induction on the number of even stairs the assertion easily follows.

By the proof of Theorem 4, we obtain the following

**Corollary 1** *Let $\mathcal{P}$ be a path polyomino such that $S/I_{\mathcal{P}}$ is pseudo-Gorenstein with $\mathcal{P} = C_1C_2\cdots C_\ell$. Then $\ell = 2r(\mathcal{P}) - 1$, namely $\ell$ is odd, and the unique facet of maximum cardinality is*

$$F = \{C_1, C_3, \dots, C_\ell\}.$$

*Example 1* We consider the stairs $\mathcal{S}_4$ of Figure 3 and $\mathcal{S}_5$ of Figure 7. The rook number of $\mathcal{S}_4$ and $\mathcal{S}_5$ is 3. The stair $\mathcal{S}_4$ has the unique facet $\{C_1, C_3, C_5\}$, while the stair $\mathcal{S}_5$ has the two facets $\{C_1, C_3, C_5\}$ and $\{C_2, C_4, C_6\}$.

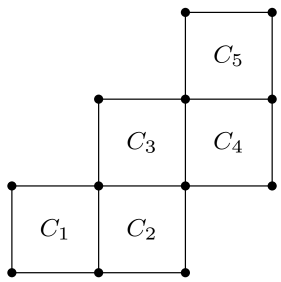



Fig. 3: The stair $\mathcal{S}_4$

## 4 Gröbner basis and levelness of path polyominoes

We start this section by defining a labelling that induces in a natural way an ordering strictly related to the cell labelling $C_1, \dots, C_\ell$ of the path. This fact, with the choice of a linear system of parameters, gives us a natural way to describe the socle of the polyomino ideal modulo the linear system of parameters.

**Notation 5** *Let $\mathcal{P}$ be a path polyomino with cells $C_1, C_2, \dots, C_\ell$. For any $i \in \{1, \dots, \ell\}$, we call $\mathcal{P}_i$ the subpath on the cells $C_1, C_2, \dots, C_i$. Then, we relabel the vertices of $V(\mathcal{P})$ as $\{a, a_1, b_1, \dots, a_\ell, b_\ell, b\}$ such that*

- *$a, a_1$ are leaf corners of $C_1$, and $b_1$ is the opposite corner of $a_1$ in $C_1$;*
- *for any $i \in \{2, \dots, \ell\}$, $a_i$ is leaf corner of $C_{i-1}$ in $\mathcal{P}_{i-1}$, and $b_i$ is the leaf corner of $C_i$ in $\mathcal{P}_i$ opposite to $a_i$.*
- *$b$ is the leaf corner of $C_\ell$ different from $b_\ell$.*

Thanks to the labelling of Notation 5, we are able to define a set of linear polynomials that we prove is a system of parameters (see Proposition 1).

**Notation 6** *Let $L = (x_a, x_{a_1} - x_{b_1}, \dots, x_{a_\ell} - x_{b_\ell}, x_b) \subseteq S$. Let $\phi : S \to R = \mathbb{K}[y_1, \dots, y_\ell]$ be the map such that $\phi(x_a) = \phi(x_b) = 0$ and $\phi(x_{a_i}) = \phi(x_{b_i}) = y_i$ for $1 \le i \le \ell$. Then it can be noted that $S/(I_{\mathcal{P}}+L) \simeq R/J_{\mathcal{P}}$, where $J_{\mathcal{P}} = \phi(I_{\mathcal{P}})$.*

To prove the levelness of $R/I_{\mathcal{P}}$ we use Gröbner basis techniques and for this aim we first focus on $\mathrm{in}(I_{\mathcal{P}})$ induced by the labelling of Notation 5. We consider the graded reverse lexicographic order $<$ on $S$ such that $x_{b_1} > x_{a_1} > \cdots > x_{b_\ell} > x_{a_\ell} > x_b > x_a$.

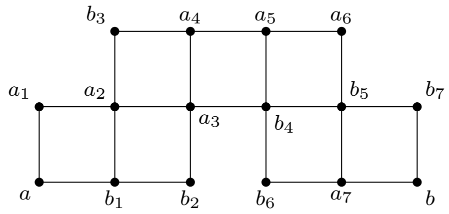



Fig. 4: Path labelled as in Notation 5

**Definition 8** We say that a path polyomino $\mathcal{P} = C_1C_2\cdots C_\ell$ has a *change of direction* at $C_i$ with $1 < i < \ell$, if $C_{i-1} \cap C_{i+1} \neq \varnothing$.

*Remark 3* From Notation 5, if there is no change of direction at $C_{i-1}$, then $C_i$ has leaf corners $a_{i+1}$ and $b_i$, and other corners $a_i$ and $b_{i-1}$ in the polyomino $\mathcal{P}_i$. The leaf corners of $C_{i+1}$ are $a_{i+2}$ and $b_{i+1}$ with the other corners either $\{a_i, a_{i+1}\}$ or $\{a_{i+1}, b_i\}$ or $\{b_{i-1}, b_i\}$ (see Figure 5).

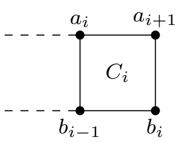



Fig. 5: The labelling of the cell $C_i$

**Lemma 2** *Let $\mathcal{P}$ be a path polyomino as in Notation 5. Then $\mathcal{M}$, the set of inner 2-minors of $\mathcal{P}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$.*

*Proof* Let $f, g \in \mathcal{M}$ be such that $\gcd(\mathrm{in}(f), \mathrm{in}(g)) \neq 1$. Let $f = f^+ - f^-$ and $g = g^+ - g^-$ with $f^+ = \mathrm{in}(f)$ and $g^+ = \mathrm{in}(g)$. We divide two cases :

1) $\gcd(f^-, g^-) \neq 1$;
2) $\gcd(f^-, g^-) = 1$;

In case 1), $f$ and $g$ are inner 2-minors contained in the same maximal interval, in particular the two intervals agree on an edge, and by definition, their $S$-polynomial reduces to 0.
In case 2), let $I_1$ and $I_2$ be the inner intervals associated to $f$ and $g$ respectively, clearly $u \in I_1 \cap I_2$ with $\gcd(f^+, g^+) = x_u$. We claim $|I_1 \cap I_2| > 1$. In fact, if $I_1 \cap I_2 = \{u\}$, then $u$ is corner of a cell $C_i$ such that there is a change of

direction at $C_i$, and $C_{i-1}$ is a cell of $I_1$ and $C_{i+1}$ is a cell of $I_2$. We say that $C_{i-1}$ has diagonal corners $u, w$ and antidiagonal corners $v, t$, while $C_{i+1}$ has diagonal corners $u, p$ and antidiagonal corners $z, q$ with $v, u, z$ lying on the same edge intervals, and $u, t, q \in C_i$. Let $c$ be the fourth corner of $C_i$. We claim that $p = a_i + 2$. We divide two cases:

i) $a_i = t$;
ii) $a_i = v$.

In case i), we have that $z = b_i$ because it is the corner of $C_i$ opposite to $t$, hence $c = a_{i+1}$, $q = b_{i+1}$ and $p = a_{i+2}$.
In case ii), we have that $c = b_i$ because it is the corner of $C_i$ opposite to $v$, hence $z = a_{i+1}$, $q = b_{i+1}$ and $p = a_{i+2}$.
In both cases, we have proved $p = a_{i+2}$, hence the claim follows, moreover the latter implies that $u$ is opposite to $a_{i+2}$ in $I_2$ that is the least variable in $I_2$ and $x_u \not| g^+$ (See Figure 6).

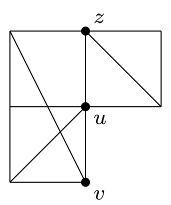



Fig. 6: The diagonals represent the leading monomials

The claim tells us that $|I_1 \cap I_2| > 1$, that is $I_1$ and $I_2$ have an edge in common, say $\{u, v\}$. Moreover let $f = x_u x_t - x_v x_w$ and $g = x_u x_z - x_p x_q$ with $p$ lying on the same edge of $u$. Without loss of generality, we assume that $u, v$ are corners of the interval $I_1$, and that $\{u, v\}$ is edge of the cell $C_i = \{u, v, p, c\}$ in the interval $I_2$, and there is a change of direction at $C_i$. We have, $I_2 \setminus \{C_i\}$ has associated inner 2-minor $g' = x_v x_z - x_c x_q$ and $I_1 \cup \{C_i\}$ has associated inner 2-minor $f' = x_t x_p - x_w x_c$. By taking the $S$-polynomial, we get

$$S(f, g) = x_z x_v x_w - x_t x_p x_q.$$

Since $w, t, q, z$ are extremal variables and $x_t | \operatorname{in}(f)$ and $x_z | \operatorname{in}(g)$, then we should identify the least variable of between $x_q$ and $x_w$.

We divide into two cases that depend on the orientation of the polyomino:

1. $\{u, v\}$ is edge of $C_{i-1}$;
2. $\{u, v\}$ is edge of $C_{i+1}$.

In case (1), $z, q$ is edge of some $C_k$ with $k > i$, and by construction we have $q = a_{k+1}$, $\operatorname{in}(S(f, g)) = x_z x_v x_w$, and $S(f, g)$ can be reduced modulo $g'$ to obtain

$$x_z x_v x_w - x_t x_p x_q - x_w g' = x_w x_c x_q - x_t x_p x_q = x_q f' \to 0.$$

In case (2), $w, t$ is edge of some $C_k$ with $k > i$, and by construction we have $w = a_{k+1}$, $\text{in}(S(f,g)) = x_t x_p x_q$, and $S(f,g)$ can be reduced modulo $f'$ to obtain

$$x_t x_p x_q - x_z x_v x_w - x_q f' = x_w x_c x_q - x_z x_v x_w = x_w g' \to 0.$$

**Lemma 3** *Let $\mathcal{P}$ be a path polyomino as in Notation 5. Then*

1. $x_{a_i} x_{a_j} \notin \text{in}(I_\mathcal{P})$ *for any* $i, j$;
2. $x_{a_i} x_{b_j} \in \text{in}(I_\mathcal{P}) \Rightarrow i \leq j$;
3. $x_{b_i} x_{b_j} \in \text{in}(I_\mathcal{P})$ *with* $i < j \Rightarrow$ *there is a change of direction at* $C_i$;
4. *Let* $f = f^+ - f^-$ *be a generator of* $I_\mathcal{P}$. *If* $f^+ = x_u x_{b_i}$ *with either* $x_u > x_{b_i}$ *or* $u = a_i$, *then* $x_{a_{i+1}} | f^-$.

*Proof* (1). Without loss of generality assume $j < i$ that is $x_{a_j} > x_{a_i}$. Let $I$ be an inner interval of $\mathcal{P}$ having opposite corners $a_i$ and $a_j$ and $u, v \in V(\mathcal{P})$. From Notation 5, we have that $i \geq j+2$ because $a_j$ and $a_{j+1}$ lie on the same edge. If $I$ is only one cell, then $i = j+2$ and $\{u, v\} = \{b_{j+1}, a_{j+1}\}$. If $I$ has cells $D_1, \dots, D_k$, then $u$ and $a_i$ are leaf corners of $D_k$, hence $u = b_{i-1}$. That is $x_u, x_v > x_{a_i}$ and $x_{a_i} x_{a_j}$ is not a leading monomial.
(2). If $i > j$, by similar arguments to (1), one can show that $x_{a_i} x_{b_j} \notin \text{in}(I_\mathcal{P})$.
(3). If $x_{b_i} x_{b_j} \in \text{in}(I_\mathcal{P})$ with $i < j$, since $b_j$ is opposite to both $b_i$ and $a_j$, then $a_j$ is on the same edge interval of $b_i$. Since $a_j$ is the leaf corner of $C_{j-1}$ and the other one is $b_{j-1}$, then $b_i$ is also opposite to $b_{j-1}$. By proceeding in this way, we have that the cell $C_{i+1}$ has opposite corners $b_i$ and $b_{i+1}$, and $v, a_{i+2}$ for some $v \in V(\mathcal{P})$. By definition, $b_i$ must be a corner of $C_i$, and from Remark 3 $a_{i+2} \notin C_i$, hence $b, v$ is an edge of $C_i$, by definition $a_{i+1}$ is a leaf corner of $C_i$, it is opposite to $b_{i+1}$ and hence is opposite to $v$, that is $v \in C_{i-1}$.
(4). By construction, $b_i$ is the leaf corner of $C_i$. Since $u$ and $b_i$ are opposite corners, then $a_{i+1}$ lies on the same edge intervals of $u$ and $b_i$, hence $x_{a_{i+1}} | f^-$.

From now to the end of the section, let $\mathcal{P} = C_1 C_2 \cdots C_\ell$ be a path polyomino with $\text{rk}(\mathcal{P}) = \ell$ and $\mathcal{C} = \{I_1, \dots, I_s\}$ be its set of cell intervals. Since a path polyomino is a simple one, by [11, Corollary 2.2], $S/I_\mathcal{P}$ is a Cohen-Macaulay domain. We now study the levelness of $S/I_\mathcal{P}$. To prove $S/I_\mathcal{P}$ level, by [24, Chapter III, Proposition 3.2], it is enough to show that for every homogeneous system of parameters $\theta_1, \dots, \theta_d$ of $S/I_\mathcal{P}$, all elements of the graded vector space $\text{Soc}(S/(I_\mathcal{P} + (\theta_1, \dots, \theta_d)))$ have the same degree. So for this, we first find homogeneous system of parameters of $S/I_\mathcal{P}$ and then study the $\text{Soc}(S/(I_\mathcal{P} + (\theta_1, \dots, \theta_d)))$.

From now on, we consider $R/J_\mathcal{P}$ of Notation 6, and therefore let "$<$" denote the graded reverse lexicographic order in $R$ induced by $y_1 > \cdots > y_\ell$.

**Proposition 1** *Let $\mathcal{P}$ be a path polyomino as in Notation 5. Then*

$$x_a, x_{a_1} - x_{b_1}, \dots, x_{a_\ell} - x_{b_\ell}, x_b$$

*is a linear system of parameters for $S/I_\mathcal{P}$.*

*Proof* Let $L = (x_a, x_{a_1} - x_{b_1}, \dots, x_{a_\ell} - x_{b_\ell}, x_b)$, and let $R$ and $J_{\mathcal{P}}$ be as in Notation 6. Let us consider the graded reverse lexicographic order "$<$" in $R$ induced by $y_1 > \cdots > y_\ell$. We claim that $y_i^2 \in \text{in}(J_{\mathcal{P}})$ for all $i = 1, \dots, \ell$. From Notation 5, we have that for any $i = 1, \dots, \ell$, $a_i$ and $b_i$ are opposite corners, and $a_{i+1}$ is on the same edge intervals of both $b_i$ and $a_i$. Hence, there exists an inner 2-minor in $I_{\mathcal{P}}$ of the form $x_{a_i}x_{b_i} - x_{a_{i+1}}x_v$ for some $v \in V(\mathcal{P})$. The image in $R/J_{\mathcal{P}}$ of such a binomial is $y_i^2 - y_{i+1}y_j$ for some $j$. We have that $y_i^2 \in \text{in}(J_{\mathcal{P}})$ which implies that $\text{in}(J_{\mathcal{P}})$ is $(y_1, \dots, y_\ell)$-primary and hence, $\text{length}_R(R/\text{in}(J_{\mathcal{P}})) < \infty$. Therefore, $\text{length}_S(S/(L+I_{\mathcal{P}})) = \text{length}_R(R/J_{\mathcal{P}}) < \infty$ and hence the assertion follows.

**Proposition 2** *Let $\mathcal{P}$ be a path polyomino. Then the generators of $J_{\mathcal{P}}$ form a Gröbner basis. Moreover,*

$$\text{in}(J_{\mathcal{P}}) = (y_iy_j : \exists\ k \in [s], C_i, C_j \in I_k).$$

*Proof* Let $f' = \phi(f), g' = \phi(g)$ be two generators of $J_{\mathcal{P}}$ with $f, g \in I_{\mathcal{P}}$ and $f = f^+ - f^-, g = g^+ - g^-$, where the map $\phi$ is defined in Notation 6.
We have to consider $S(f', g')$ in the cases $\gcd(\text{in}(f'), \text{in}(g')) \neq 1$, that is $\deg(\gcd(\text{in}(f'), \text{in}(g'))) \in \{1, 2\}$.
If $\deg(\gcd(\text{in}(f')\,\text{in}(g'))) = 2$, namely $\text{in}(f') = \text{in}(g') = y_iy_j$ with $i < j$, then from Lemma 3, $x_{b_j} \mid \text{in}(f)$ and $x_{b_j} \mid \text{in}(g)$, that is without loss of generality we assume that $\text{in}(f) = x_{a_i}x_{b_j}$ and $\text{in}(g) = x_{b_i}x_{b_j}$ but this cannot happen because $a_i$ and $b_i$ are opposite corners and the polyomino is thin.
If $\gcd(\text{in}(f'), \text{in}(g')) = \phi(\gcd(\text{in}(f), \text{in}(g)))$, then $S(f', g') = \phi(S(f, g)) \to 0$.
If $\gcd(\text{in}(f'), \text{in}(g')) \neq \phi(\gcd(\text{in}(f), \text{in}(g))) = 1$, then $x_{a_i} \mid \text{in}(f)$ and $x_{b_i} \mid \text{in}(g)$. That is $\text{in}(f) = x_{a_i}x_{b_j}$ for some $j > i$ (if $j = i$ we are in the previous case) and $\text{in}(g) = x_ux_{b_i}$ with $u \in V(\mathcal{P})$. From Lemma 3.(4), we get that

$$f = x_{a_i}x_{b_j} - x_vx_{a_{j+1}},$$

for some $v$ in $V(\mathcal{P})$ with $x_v > x_{b_i}, x_{a_i}$. One can observe that since $a_i$ and $b_j$ are opposite corners as well as $a_i$ and $b_i$, then there is no change of direction at $C_i$, and it can not happen that $u = b_k$ with $k > i$. Moreover, from Lemma 3.(1), also the case $u = a_k$ with $k > i$ is not possible. Hence, we can only have $u = a_k$ or $u = b_k$ for some $k < i$. From Lemma 3.(4), we get that $x_{a_{i+1}} | g^-$ and

$$g = x_ux_{b_i} - x_wx_{a_{i+1}},$$

for some $w$ in $V(\mathcal{P})$. By construction, $b_j$ lies on the same edge interval of $b_i$ as well as $u$ lies on the same edge interval of $a_i$, hence from $j > i$ we get that $h_1 = x_{a_{i+1}}x_{b_j} - x_{b_i}x_{a_{j+1}}$ and $h_2 = x_ux_v - x_wx_{a_i}$ are generators of $I_{\mathcal{P}}$. We have

$$f' = y_iy_j - \phi(x_v)y_{j+1} \quad g' = \phi(x_u)y_i - \phi(x_w)y_{i+1}$$

$$h_1' = \phi(h_1) = y_{i+1}y_j - y_iy_{j+1}, \quad h_2' = \phi(h_2) = \phi(x_u)\phi(x_v) - \phi(x_w)y_i$$

hence

$$S(f', g') = y_j\phi(x_w)y_{i+1} - \phi(x_u)\phi(x_v)y_{j+1}$$

by reducing modulo $h_1'$ we get

$$S(f', g') = y_iy_{j+1}\phi(x_w) - \phi(x_u)\phi(x_v)y_{j+1} = y_{j+1}h_2'.$$

*Remark 4* Let $\mathcal{P}$ be a path polyomino. Then $y_iy_j \in \text{in}(J_{\mathcal{P}})$ with $j \leq i$ if and only if the cells $C_i$ and $C_j$ are attacking.

**Corollary 2** *The standard monomials in $R/J_{\mathcal{P}}$, namely the monomials not in $\text{in}(J_{\mathcal{P}})$, are the squarefree monomials*

$$u = y_{i_1}y_{i_2}\cdots y_{i_{s'}} \text{ with } i_1 < i_2 < \cdots < i_{s'},$$

*where $s'$-rooks are placed in the non-attacking cells $C_{i_1}, \dots, C_{i_{s'}}$ of $\mathcal{P}$.*

*Proof* If two rooks are placed in the non-attacking cells $C_{i_j}$, $C_{i_k}$ with $i_j \leq i_k$, then $C_{i_j}$ and $C_{i_k}$ do not belong to the same interval. By Remark 4, $y_{i_j}y_{i_k} \notin \text{in}(J_{\mathcal{P}})$ for $1 \leq j \leq s'-1$. Therefore, $u \notin \text{in}(J_{\mathcal{P}})$.

Suppose two rooks are placed in the cells $C_{i_j}$, $C_{i_k}$ with $i_j \leq i_k$ so that they can attack each other and $y_{i_j}y_{i_k}$ divides our monomial. Then there exist $m$ such that $i_j, i_k \in I_m$. Then it is reducible by

$$y_{i_j}y_{i_k} - y_{i_j-1}y_{i_k+1},$$

hence our monomial is not in the standard form. Contradiction.

Now we write all the monomial generators of $J_{\mathcal{P}}$ which come from the first cell interval and last cell interval. Let $J_0$ and $J_s$ be the subideals of $J_{\mathcal{P}}$ generated by monomials coming from the first and last cell intervals respectively. Then

$$J_0 = (y_iy_j : 1 \leq i \leq j \leq l_1 - i + 1) \text{ and}$$
$$J_s = (y_iy_j : \ell \geq i \geq j \geq 2\ell - i - (l_s - 1)).$$

We also recall that the set of configurations of pairwise non-attacking rooks is a simplicial complex denoted by $\mathcal{R}_{\mathcal{P}}$. Moreover for $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ we set

$$y_F = \prod_{C_i \in F} y_i.$$

**Lemma 4** *Let $\mathcal{P}$ be a path polyomino. Then*

$$\text{Soc}(R/\text{in}(J_{\mathcal{P}})) = (y_F \mid F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})).$$

*Proof* The socle of a monomial ideal is a monomial ideal. Let $u$ be a monomial of $R \setminus J_{\mathcal{P}}$. Since $y_i^2 \in \text{in}(J_{\mathcal{P}})$, then $u$ is squarefree and $u = y_F$ for some $F \in \mathcal{R}_{\mathcal{P}}$. Assume that $F$ is not maximal, then there exists $F' \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ such that $F \subset F'$ and let $C_i \in F' \setminus F$. Since $C_i$ attacks no cell in $F$, then $uy_i \notin \text{in}(J_{\mathcal{P}})$, contradicting the hypothesis.
Conversely, assume that $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ and $u = y_F$. Now we prove that for all $i \in \{1, \dots, \text{rk}\,\mathcal{P}\}$, $uy_i \in \text{in}(J_{\mathcal{P}})$. Fix $i \in \{1, \dots, \text{rk}\,\mathcal{P}\}$. Since $F$ is maximal, then the cell $C_i$ is attacked by a cell $C_j$ in $F$, hence it follows from Remark 4 that $y_iy_j \in \text{in}(J_{\mathcal{P}})$.

It is interesting to observe that the facets of maximum cardinality of $\mathcal{F}(\mathcal{R}_{\mathcal{P}})$ induces also maximum degree elements (monomials) in $\text{Soc}(R/J_{\mathcal{P}})$, as pointed out in Corollary 4.

**Corollary 3** *Let $\mathcal{P}$ be a path polyomino. Then $R/\operatorname{in}(J_{\mathcal{P}})$ is level if and only if $\mathcal{R}_{\mathcal{P}}$ is pure.*

Since $\operatorname{in}(J_{\mathcal{P}})$ level implies $J_{\mathcal{P}}$ level and thanks to the relations between $\operatorname{Soc}(S/I_{\mathcal{P}})$ and $\operatorname{Soc}(R/J_{\mathcal{P}})$, if $R_{\mathcal{P}}$ is pure then also $I_{\mathcal{P}}$ is level.

By using Theorem 1, we obtain the following characterization of path polyominoes having $R/\operatorname{in}(J_{\mathcal{P}})$ level.

**Theorem 7** *Let $\mathcal{P}$ be a path polyomino with $\mathcal{C} = \{I_1, I_2, \dots, I_s\}$. Then $R/\operatorname{in}(J_{\mathcal{P}})$ is level with $r(\mathcal{P}) = d$ if and only if the followings hold*

1. *$s = 2d - 1$.*
2. *for any $2 \leq k \leq s - 1$ we have*

$$\begin{cases} l_k > 2 & \text{if } k \text{ is odd} \\ l_k = 2 & \text{if } k \text{ is even} \end{cases}$$

*Proof* According to Corollary 3, $R/\operatorname{in}(J_{\mathcal{P}})$ is level if and only if $\mathcal{R}_{\mathcal{P}}$ is pure, that is if and only if $\mathcal{P}$ admits a super partition $\mathcal{A}$. We proceed by induction on $d$. We start from $d = 2$. $\mathcal{A}$ contains two intervals, since $I_1 \in \mathcal{A}$, then $I_2 \notin \mathcal{A}$ and $s = 3$. If $I_2$ contains single cell then $I_2 \in \mathcal{A}$, hence $|I_2| = 2$. We assume $d > 2$ and the thesis holds true for $d - 1$. From similar arguments, one gets that $|I_2| = 2$ and that the polyomino $\mathcal{P}' = \mathcal{P} \setminus I_1$ is super partitionable, with partition $\mathcal{A}' = \mathcal{A} \setminus I_1$. Hence $s - 2 = 2(d - 1) - 1$, that is $s = 2d - 1$ and $l_k > 2$ for an odd $k \in \{4, \dots, s - 1\}$ and $l_k = 2$ for even $k \in \{4, \dots, s - 1\}$. We prove that $l_3 > 2$. If $l_3 = 2$, then $I_3 = [C_i, C_{i+1}]$ and $C_{i-1} \in I_1$, $C_{i+2} \in I_5$ with $C_{i-1}$ (resp. $C_{i+2}$) attacking $C_i$ (resp. $C_{i+1}$). That is $I_3$ is an embedded interval. Contradiction.

We now study the levelness of $R/J_{\mathcal{P}}$ with the help of the following:

**Lemma 5** *Let $I \subseteq R$ be an ideal with $\dim(R/I) = 0$. Then for any monomial ordering $<$, one has*

$$\operatorname{in}(\operatorname{Soc}(R/I)) \subseteq \operatorname{Soc}(R/\operatorname{in}(I)).$$

*Proof* Let $u \in \operatorname{in}(\operatorname{Soc}(R/I))$, then there exists $g$ with $\operatorname{in}(g) < u$ such that $f = u + g$ and $fy_i \in I$ for all $1 \leq i \leq n$. Then $uy_i \in \operatorname{in}(I)$ for all $1 \leq i \leq n$, hence $u \in \operatorname{Soc}(R/\operatorname{in}(I))$.

**Lemma 6** *Let $\mathcal{P}$ be a path polyomino, let $F$ be a facet of $\mathcal{R}_{\mathcal{P}}$. If there exists $k$ such that $C_k \in F$ and*

$$F' = (F \setminus \{C_k\}) \cup \{C_{k-1}, C_{k+1}\}$$

*is a facet of $\mathcal{R}_{\mathcal{P}}$, then $y_F \notin \operatorname{in}(\operatorname{Soc}(R/J_{\mathcal{P}}))$.*

*Proof* We consider $y_k y_F$. The relation $y_k^2 - y_{k-1}y_{k+1} \in J_{\mathcal{P}}$, that is $y_k y_F$ reduces to $y_{F'}$. By Corollary 2, $y_{F'} \notin \operatorname{in}(J_{\mathcal{P}})$. Hence $y_F \notin \operatorname{in}(\operatorname{Soc}(R/J_{\mathcal{P}}))$.

*Example 2* The stair $\mathcal{S}_5$ (see Fig (7)) is such that $R/J_{\mathcal{P}}$ is level but $R/\operatorname{in}(J_{\mathcal{P}})$ is not. In fact

$$\mathcal{F}(\mathcal{R}_{\mathcal{P}})) = \{\{C_1, C_3, C_5\}, \{C_1, C_3, C_6\}, \{C_1, C_4, C_6\}, \{C_2, C_4, C_6\}, \{C_2, C_5\}\}.$$

Hence $\mathcal{R}_{\mathcal{P}}$ is not pure and by Corollary 3, $R/\operatorname{in}(J_{\mathcal{P}})$ is not level. If $R/J_{\mathcal{P}}$ is not level there exists $f \in \operatorname{Soc}(R/J_{\mathcal{P}})$ whose degree is strictly less than the rook number of $\mathcal{P}$. Moreover, $\operatorname{in}(f) = y_F = y_2y_5$ (see Lemma 4 and Lemma 5). But $F' = (F \setminus \{C_2\}) \cup \{C_1, C_3\} \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ and by Lemma 6, $y_F \notin \operatorname{in}(\operatorname{Soc}(R/J_{\mathcal{P}}))$.

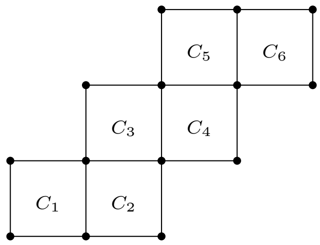



Fig. 7: The stair $\mathcal{S}_5$

**Proposition 3** *Let $\mathcal{P}$ be a path polyomino. If $\operatorname{reg}(S/I_{\mathcal{P}}) = t$, namely $r(\mathcal{P}) = t$, then $\mathfrak{n}^{t+1} \subseteq J_{\mathcal{P}}$, where $\mathfrak{n} = (y_1, \dots, y_\ell)$ is the homogeneous maximal ideal in $R$.*

*Proof* Since $\mathcal{P}$ is a simple thin polyomino, by [20, Corollary 3.16], $\operatorname{reg}(S/I_{\mathcal{P}}) = t = r(\mathcal{P})$, where $r(\mathcal{P})$ is the rook number. Let $u = y_{i_1} \cdots y_{i_{t+1}} \in \mathfrak{n}^{t+1}$ be any element. Therefore by Corollary 2, $u$ is not a standard monomial in $R/J_{\mathcal{P}}$. This implies that there are at least two cells $C_{i_j}, C_{i_k}$ in the attacking positions with $i_j \leq i_k$. If the two cells belong to the first cell interval (resp. the last cell interval) $u \in J_0$ (resp. $u \in J_s$). Otherwise, there exists a binomial $y_{i_j}y_{i_k} - y_{i_j-1}y_{i_k+1} \in J_{\mathcal{P}}$, to reduce $u$ in $u_1$.

Moreover, since $u_1$ has degree $t+1$, as $u$, there are two cells in attacking position induced by the support of $u_1$, too. Then we apply the same procedure again on $u_1$ to get $u_2$. We continue this process until we get $u_l \in J_0$ or $u_l \in J_s$ so that $u \in J_{\mathcal{P}}$. Thus, we have $\mathfrak{n}^{t+1} \subseteq J_{\mathcal{P}}$.

**Corollary 4** *Let $\mathcal{P}$ be a polyomino, let $F$ be a facet of $\mathcal{R}_{\mathcal{P}}$ with $|F| = r(\mathcal{P})$. Then $y_F \in \operatorname{Soc}(R/J_{\mathcal{P}})$.*

**Corollary 5** *Let $\mathcal{P}$ be a path polyomino with $\operatorname{reg}(S/I_{\mathcal{P}}) = 2$. Then $S/I_{\mathcal{P}}$ is level.*

*Proof* Since $\operatorname{reg}(S/I_{\mathcal{P}}) = 2$, $\mathcal{P}$ is not a cell interval. Then it follows from the description of monomial generators $J_0 \cup J_s$ of $J_{\mathcal{P}}$ that $y_1y_\ell \notin J_{\mathcal{P}}$ and $y_i^2 \notin J_{\mathcal{P}}$ for $2 \leq i \leq \ell - 1$. This implies that $y_i \notin \operatorname{Soc}(R/J_{\mathcal{P}})$ for $1 \leq i \leq \ell$. By Proposition 3, $\operatorname{Soc}(R/J_{\mathcal{P}})$ has generator of degree at most 2 and hence $\operatorname{Soc}(R/J_{\mathcal{P}})$ is generated in degree 2 only. Therefore, by [24, Chapter III, Proposition 3.2], $S/I_{\mathcal{P}}$ is level.

Now we give another class of path polyominoes which are level.

**Theorem 8** *Let $\mathcal{P}$ be a path polyomino with $l_1, l_s \geq 2$, and $l_i \geq 3$ for $2 \leq i \leq s-1$. Then $S/I_{\mathcal{P}}$ is level.*

*Proof* We prove that $S/I_{\mathcal{P}}$ is level, by proving that $\mathrm{Soc}(R/J_{\mathcal{P}})$ is generated in degree $r(\mathcal{P})$. From Lemma 5 and Corollary 4 , we have

$$\mathrm{in}(\mathrm{Soc}(R/J_{\mathcal{P}})) \subseteq \mathrm{Soc}(R/\,\mathrm{in}(J_{\mathcal{P}})),$$

and for any $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ with $|F| = r(\mathcal{P})$, $y_F \in \mathrm{in}(\mathrm{Soc}(J_{\mathcal{P}}))$. Hence it is sufficient to show that for any $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ with $|F| < r(\mathcal{P})$ it holds $y_F \notin \mathrm{in}(\mathrm{Soc}(R/J_{\mathcal{P}}))$. By construction any interval of $\mathcal{P}$ has a single cell and $r(\mathcal{P}) = s$. Since $|F| < r(\mathcal{P})$, then $F$ contains a non-single cell $C_k$. Since $l_k > 2$, $C_{k-1}$ and $C_{k+1}$ are single cells, hence

$$F' = (F \setminus \{C_k\}) \cup \{C_{k-1}, C_{k+1}\} \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$$

and from Lemma 6 the claim follows.

Now we need some technical results to classify the levelness of $S/I_{\mathcal{P}}$.

**Lemma 7** *Let $\mathcal{P}$ be a path polyomino. Let $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ be such that there exists even $h \geq 2$, $k$, and $\{C_k, C_{k+2}, C_{k+4} \ldots, C_{k+h-2}, C_{k+h}\} \subset F$ (that is $C_k, \ldots, C_{k+h}$ lie on a stair).*

1. *Assume that $F' = (F \setminus \{C_k, C_{k+h}\}) \cup \{C_{k-1}, C_{k+h+1}\} \in \mathcal{R}_{\mathcal{P}}$. Then for any $j \in \{k, k+1, \ldots, k+h\}$*

$$y_j y_F = y_j y_{F'} \mod J_{\mathcal{P}}.$$

2. *Assume that $F'' = (F \setminus \{C_k\}) \cup \{C_{k-1}\} \in \mathcal{R}_{\mathcal{P}}$. Then it follows*

$$y_{k+h+1} y_F = y_{k+h+2} y_{F''} \mod J_{\mathcal{P}}$$

3. *Assume that $F''' = (F \setminus \{C_{k+h}\}) \cup \{C_{k+h+1}\} \in \mathcal{R}_{\mathcal{P}}$.Then it follows*

$$y_{k-1} y_F = y_{k-2} y_{F'''} \mod J_{\mathcal{P}}.$$

*Proof* Without loss of generality, we assume that $F = \{C_k, C_{k+2}, \ldots, C_{k+h}\}$.
(1). We proceed by induction on $h$. Moreover, to simplify the notation, all the equalities are equalities modulo $J_{\mathcal{P}}$.
Let $h = 2$, then $F = \{C_k, C_{k+2}\}$. We prove that $y_j y_k y_{k+2} = y_j y_{k-1} y_{k+3}$ for $j = k, k+1, k+2$.

- If $j = k$, then $y_k^2 y_{k+2} = y_{k-1} y_{k+1} y_{k+2} = y_{k-1} y_k y_{k+3}$.
- If $j = k+1$, then $y_k y_{k+1} y_{k+2} = y_{k-1} y_{k+2}^2 = y_{k-1} y_{k+1} y_{k+3}$.
- If $j = k+2$, then $y_k y_{k+2}^2 = y_k y_{k+1} y_{k+3} = y_{k-1} y_{k+2} y_{k+3}$.

Hence the assertion follows. We assume that $h > 2$ and the thesis holds true for $h-2$, that is the thesis holds true for any facet of $h/2$ cells of the form $\{D_t, D_{t+2} \dots, D_{t+h-2}\}$. We consider

$$F = \{C_k, C_{k+2}, \dots, C_{k+h-2}, C_{k+h}\}, \quad F' = \{C_{k-1}, C_{k+2}, \dots C_{k+h-2}, C_{k+h+1}\}.$$

We prove that for any $j \in \{k, \dots, k+h\}$, $y_j y_F = y_j y_{F'}$ modulo $J_{\mathcal{P}}$. We consider

$$F_1 = F \setminus \{C_{k+h}\}, \text{ and } F_1' = (F_1 \setminus \{C_k, C_{k+h-2}\}) \cup \{C_{k-1}, C_{k+h-1}\},$$

and $F_1$, $F_1'$ are faces because $F$ and $F'$ are faces by hypothesis. By inductive hypothesis, we have that for any $j \in \{k, \dots, k+h-2\}$, $y_j y_{F_1} = y_j y_{F_1'}$ and hence

$$y_j y_F = y_j y_{F_1} y_{k+h} = y_j y_{F_1'} y_{k+h} = y_j \frac{y_{F_1'}}{y_{k+h-1}} y_{k+h-1} y_{k+h} =$$
$$= y_j y_{k-1} y_{k+2} \cdots y_{k+h-4} y_{k+h-2} y_{k+h+1} = y_j y_{F'}.$$

We are left with the cases $j \in \{k+h-1, k+h\}$. If $j = k+h-1$, then $y_{k+h-1} y_F$ is equal to

$$y_k y_{k+2} \cdots y_{k+h-2} y_{k+h-1} y_{k+h} = y_k y_{k+2} \cdots y_{k+h-2}^2 y_{k+h+1} = (y_{F_1} y_{k+h-2}) y_{k+h+1}.$$

We apply the inductive hypothesis and we get that $y_{k+h-2} y_{F_1} = y_{k+h-2} y_{F_1'}$, that is

$$(y_{k+h-2} y_{F_1}) y_{k+h+1} = y_{k+h-2} y_{k-1} y_{k+2} \cdots y_{k+h-4} y_{k+h-1} y_{k+h+1} = y_{k+h-1} y_{F'}.$$

If $j = k+h$, then $y_{k+h} y_F$ is equal to

$$y_k y_{k+2} \cdots y_{k+h-2} y_{k+h}^2 = y_k y_{k+2} \cdots y_{k+h-2} y_{k+h-1} y_{k+h+1} = (y_{F_1} y_{k+h-1}) y_{k+h+1}.$$

We apply the inductive hypothesis and we get that $y_{k+h-1} y_{F_1} = y_{k+h-1} y_{F_1'}$, that is

$$(y_{k+h-1} y_{F_1}) y_{k+h+1} = y_{k+h-1} y_{k-1} y_{k+2} \cdots y_{k+h-4} y_{k+h-1} y_{k+h+1} =$$
$$y_{k-1} y_{k+2} \cdots y_{k+h-4} y_{k+h-1}^2 y_{k+h+1} = y_{k-1} y_{k+2} \cdots y_{k+h-4} y_{k+h-2} y_{k+h} y_{k+h+1} =$$
$$y_{k+h} y_{F'}.$$

Since $y_j y_F = y_j y_{F'}$ modulo $J_{\mathcal{P}}$, the assertion follows.

(2). We proceed by induction on $h$. If $h = 2$, then $F = \{C_k, C_{k+2}\}$. We prove that $y_{k+3} y_k y_{k+2} = y_{k-1} y_{k+2} y_{k+4}$. We have,

$$y_k y_{k+2} y_{k+3} = y_k y_{k+1} y_{k+4} = y_{k-1} y_{k+2} y_{k+4}.$$

We assume that $h > 2$ and that the thesis holds true for any facet of $h/2$ cells of the form $\{D_t, D_{t+2} \dots, D_{t+h-2}\}$. We consider

$$F = \{C_k, C_{k+2}, \dots, C_{k+h}\}, \quad F'' = \{C_{k-1}, C_{k+2}, \dots C_{k+h}\}.$$

We prove that $y_{k+h+1}y_F = y_{k+h+2}y_{F'}$. We have

$$y_k y_{k+2} \cdots y_{k+h-2} y_{k+h} y_{k+h+1} = y_k y_{k+2} \cdots y_{k+h-2} y_{k+h-1} y_{k+h+2}$$

by applying the inductive hypothesis on $\{C_k, C_{k+2}, \dots, C_{k+h-2}\}$, we get

$$y_k y_{k+2} \cdots y_{k+h-2} y_{k+h-1} y_{k+h+2} = y_{k-1} y_{k+2} \cdots y_{k+h-2} y_{k+h} y_{k+h+2} = y_{k+h+2} y_{F''},$$

as desired.
(3). Similarly to (2).

**Lemma 8** *Let $\mathcal{P}$ be a path polyomino. Let $F \in \mathcal{F}(\mathcal{R}_\mathcal{P})$ be such that there exists $h \geq 2$, $\ell \geq 2$, $k$, and $\{C_k, C_{k+2}, \dots, C_{k+h}, C_{k+h+3}, C_{k+h+5}, \dots, C_{k+h+\ell+3}\} \subset F$ with $C_k, \dots, C_{k+h+\ell+3}$ lying on a stair and such that $F' = (F \setminus \{C_k, C_{k+h+\ell+3}\}) \cup \{C_{k-1}, C_{k+h+\ell+4}\} \in \mathcal{R}_\mathcal{P}$. Then for $j \in \{k+h+1, k+h+2\}$*

$$y_j(y_F - y_{F'}) \in J_\mathcal{P}.$$

*Proof* We assume that the equalities are modulo $J_\mathcal{P}$. We prove that for $j \in \{k+h+1, k+h+2\}$, $y_j y_F = y_j y_{F'}$. Let $j = k+h+1$, we obtain that $y_j y_F$ is

$$y_k y_{k+2} \cdots y_{k+h} (y_{k+h+1}) y_{k+h+3} y_{k+h+5} \cdots y_{k+h+l+3}.$$

We apply Lemma 7.(2) to get

$$y_k y_{k+2} \cdots y_{k+h} y_{k+h+1} = y_{k-1} y_{k+2} \cdots y_{k+h} y_{k+h+2},$$

and Lemma 7.(3) to get

$$y_{k+h+2} y_{k+h+3} y_{k+h+5} \cdots y_{k+h+l+3} = y_{k+h+1} y_{k+h+3} y_{k+h+5} \cdots y_{k+h+l+4}.$$

Hence the thesis follows. The case $j = k+h+2$ similarly follows by applying Lemma 7.(3) and Lemma 7.(2).

**Proposition 4** *Let $\mathcal{P} = \mathcal{S}_\lambda$ such that $\lambda = 4, 6$ or $\lambda \geq 8$. Then $S/I_\mathcal{P}$ is not level.*

*Proof* If $\mathcal{P}$ is the stair $\mathcal{S}_\lambda$, then $|\mathcal{P}| = \lambda + 1$.
If $\lambda$ is even and $\lambda \geq 4$, then $\lambda = 2k$ and $r(\mathcal{P}) = k + 1$. We consider $F = \{C_2, C_4, \dots, C_\lambda\} \in \mathcal{F}(\mathcal{R}_\mathcal{P})$ and has cardinality $k$. Let $F' = \{C_1, C_4, C_6, \dots, C_{\lambda-2}, C_{\lambda+1}\}$. We prove that $y_F - y_{F'} \in \mathrm{Soc}(R/J_\mathcal{P})$, namely that for any $j \in \{1, \dots, \lambda+1\}$

$$y_j(y_F - y_{F'}) \in J_\mathcal{P}.$$

For $j = 1, \lambda+1$, it easily follows since $y_1^2, y_1 y_2 \in J_0 \subseteq J_\mathcal{P}$ and $y_\lambda y_{\lambda+1}, y_{\lambda+1}^2 \in J_\lambda \subseteq J_\mathcal{P}$, while for $j \in \{2, 3, \dots, \lambda\}$ this is the case of Lemma 7.(1). Hence, the first part of the assertion follows.

If $\lambda$ is odd and $\lambda \geq 9$, then $\lambda = 2m - 1$ and $r(\mathcal{P}) = m$. We first deal with the case $m$ odd. We consider

$$F = \{C_2, C_4\} \cup \{C_{2k-1} : 4 \leq k \leq m\} \text{ and } F' = \{C_1, C_5\} \cup \{C_{2k-1} : 4 \leq k \leq m\}.$$

Since $m$ is odd, then $m-3=|\{C_{2k-1}:4\leq k\leq m\}|$ is even and $m-3=2t$ for some $t$. We further consider

$$F''=\{C_{2k}:k\in\{1,2,\dots,m-t-1,m-t+1,\dots,m\}\}$$

$$F'''=\{C_1\}\cup\{C_{2k}:k\in\{2,\dots,m-t-2\}\}\cup$$
$$\cup\{C_{2(m-t)-1}\}\cup\{C_{2k}:k\in\{m-t+1,\dots,m\}\}.$$

We claim that for any $j\in\{1,\dots,\lambda+1\}$ we have

$$y_j(y_F-y_{F'}-y_{F''}+y_{F'''})\in J_{\mathcal{P}}.$$

From now on we deal with the cases $j=2h-1$ for $h=1,\dots,m$. The case $j=2h$ equivalently follows.
**Claim:** Let

$$G=\{C_{2k-1}:4\leq k\leq m\}.$$

For $j=2h-1$ with $4\leq h\leq m$, we claim that

$$y_{2h-1}y_G=y_{2h-1}y_H \mod J_{\mathcal{P}}, \tag{1}$$

where, given

$$G_h=\{C_{2k}:3\leq k\leq h-1\}\cup\{C_{2k-1}:h+1\leq k\leq m-h+3\}\cup$$
$$\cup\{C_{2k}:m-h+4\leq k\leq m\},$$

and

$$H=\begin{cases}G_h & \text{if } h\in\{4,\dots,\frac{m-3}{2}+3\}\\ G_{m-h+4} & \text{if } h\in\{\frac{m-3}{2}+4,\dots,m\}\end{cases}.$$

*Proof of the claim:* If $4\leq h\leq m$, then from Lemma 7.(1) we have

$$y_{2h-1}y_G=y_{2h-1}y_{G'} \mod J_{\mathcal{P}}$$

where $G'=\{C_6\}\cup\{C_{2k-1}:5\leq k\leq m-1\}\cup\{C_{2m}\}=G_4$. Moreover, if $5\leq h\leq m-1$, then we can do the same argument for $\{C_{2k-1}:5\leq k\leq m-1\}$. We can repeat the same argument until we get

$$y_{2h-1}y_G=y_{2h-1}y_{G_h} \mod J_{\mathcal{P}}, \tag{2}$$

where

$$G_h=\{C_{2k}:3\leq k\leq h-1\}\cup\{C_{2k-1}:h+1\leq k\leq m-h+3\}\cup$$
$$\cup\{C_{2k}:m-h+4\leq k\leq m\}.$$

This set makes sense for $h\in\{4,\dots,\frac{m-3}{2}+3=t+3\}$. If $h\in\{t+4,\dots,m\}$, then $h'=m-h+4\in\{4,\dots,t+3\}$, and

$$y_{2h-1}y_G=y_{2h-1}y_{G_{h'}} \mod J_{\mathcal{P}}.$$

This proves the claim.

We want to compute $y_j y_F$, $y_j y_{F'}$, $y_j y_{F''}$, $y_j y_{F'''}$ for $j = 1, \dots, m$. Given $j \in \{2h-1, 2h\}$, for $4 \leq h \leq m$ we use Equation (2) to get

$$y_j y_F = y_j y_2 y_4 y_G = y_j y_2 y_4 y_{G_h}$$

$$y_j y_{F'} = y_j y_1 y_5 y_G = y_j y_1 y_5 y_{G_h}.$$

We divide into three cases:

1. $h < t+3$;
2. $h = t+3$;
3. $h > t+3$.

(1). We have that for any $h = \{2, 3 \dots, m-t-1 = t+2\}$, from Lemma 7.(1) applied to the set $\{C_{2k} : k \in \{1, 2, \dots, m-t-1\}\}$, we have $y_j(y_{F''} - y_{F'''}) \in J_{\mathcal{P}}$. Hence we only should control that we have $y_j(y_F - y_{F'}) \in J_{\mathcal{P}}$. Moreover from Lemma 7 and Lemma 8 for $j = 2, \dots, 6$, $y_j(y_F - y_{F'}) \in J_{\mathcal{P}}$. Since

$$h < t+3 \quad \Rightarrow \quad 2h < m+3 \quad \Rightarrow \quad h < m-h+3 \quad \Rightarrow \quad h+1 \leq m-h+3,$$

then $\{C_{2k-1} : h+1 \leq k \leq m-h+3\} \neq \varnothing$. Then $C_{2h-2}$ and $C_{2h+1}$ have a 3 step difference, that is we are in the hypotheses of Lemma 8, and from Lemma 8, we get

$$y_j y_2 y_4 y_{G_h} = y_j y_1 y_4 \frac{y_{G_h}}{y_{2(m-h+3)-1}} y_{2(m-h+3)}.$$

On the other hand, we compute

$$y_j y_{F'} = y_j y_1 y_5 y_G = y_j y_1 y_5 y_{G_h}.$$

Since $\Big(\prod_{k=3}^{h-1} y_{2k}\Big) | y_{G_h}$, then we apply Lemma 7.(3) to

$$y_5 \Big(\prod_{k=3}^{h-1} y_{2k}\Big) = y_4 \Big(\prod_{k=3}^{h-2} y_{2k}\Big) y_{2h-1}$$

and hence

$$y_j y_1 y_5 y_{G_h} = y_j y_1 y_4 \frac{y_{G_h}}{y_{2h-2}} y_{2h-1}.$$

Since $\Big(\prod_{k=h+1}^{m-h+3} y_{2k-1}\Big) | y_{G_h}$, then $\Big(\prod_{k=h}^{m-h+3} y_{2k-1}\Big) | y_{2h-1} y_{G_h}$ and since $j \in \{2h-1, 2h\}$, we apply Lemma 7.(1) to get

$$y_j \Big(\prod_{k=h}^{m-h+3} y_{2k-1}\Big) = y_j y_{2h-2} \Big(\prod_{k=h+1}^{m-h+2} y_{2k-1}\Big) y_{2(m-h+3)},$$

hence

$$y_j y_1 y_4 \frac{y_{G_h}}{y_{2h-2}} y_{2h-1} = y_j y_1 y_4 \frac{y_{G_h}}{y_{2h-2} y_{2(m-h+3)-1}} y_{2h-2} y_{2(m-h+3)} =$$

$$= y_j y_1 y_4 \frac{y_{G_h}}{y_{2(m-h+3)-1}} y_{2(m-h+3)}.$$

Hence $y_j(y_F - y_{F'}) = 0$.
(2) In the case $h = t+3$, we have $m-h+4 = m-t+1 = h+1$ and hence $h-1 = m-t-1$ and $\{C_{2k-1} : h+1 \leq k \leq m-h+3\} = \varnothing$. Therefore,

$$G_{m-t} = \{C_{2k} : 3 \leq k \leq m-t-1\} \cup \{C_{2k} : m-t+1 \leq k \leq m\}$$

and

$$y_j y_F = y_j y_2 y_4 y_{G_{m-t}} = y_j y_{F''}.$$

Similarly, from Lemma 7.(3) we have $y_5 \prod_{k=3}^{m-t-1} y_{2k} = y_4 \Big( \prod_{k=3}^{m-t-2} y_{2k} \Big) y_{2(m-t)-1}$ and

$$y_j y_{F'} = y_j y_1 y_5 y_{G_{m-t}} = y_1 y_4 \frac{y_{G_{m-t}}}{y_{2(m-t)-2}} y_{2(m-t)-1} = y_j y_{F'''},$$

That is $y_j(y_F - y_{F'} - y_{F''} + y_{F'''}) = 0$ modulo $J_{\mathcal{P}}$.
(3) In the case $h > t+3$, we set $h' = m-h+4$, we have

$$G_{h'} = \{C_{2k} : 3 \leq k \leq h'-1\} \cup \{C_{2k-1} : h+1 \leq k \leq m-h+3\} \cup$$

$$\cup \{C_{2k} : m-h'+4 \leq k \leq m\},$$

hence and from Lemma 7.(3)

$$y_j \Big( \prod_{k=m-h'+4}^{m-1} y_{2k} \Big) = y_j \Big( \prod_{k=m-h'+4}^{m-2} y_{2k} \Big) y_{2m-1}$$

and hence

$$y_j y_2 y_4 y_{G_{h'}} = 0$$

because $y_{2m-1} y_{2m} | y_j y_2 y_4 y_{G_{h'}}$. The same can be proved for $y_{F'}$ and since $h \geq m-t+1$, then also for $y_{F''}$ and $y_{F'''}$. This concludes the case $m$ odd.
If $m$ is even, then similar arguments hold, with

$$F = \{C_2, C_4\} \cup \{C_{2k-1} : 4 \leq k \leq m-1\} \cup \{C_{2m}\}$$

$$F' = \{C_1, C_5\} \cup \{C_{2k-1} : 4 \leq k \leq m-1\} \cup \{C_{2m}\},$$

and given $m-4 = |\{C_{2k-1} : 4 \leq k \leq m-1\}| = 2t$, we have

$$F'' = \{C_{2k} : k \in \{1, 2, \ldots, m-t-1, m-t+1, \ldots, m\}\}$$

$$F''' = \{C_1\} \cup \{C_{2k} : k \in \{2, \ldots, m-t-3\}\} \cup \{C_{2(m-t-2)-1}\} \cup$$

$$\cup \{C_{2k} : k \in \{m-t, \ldots, m\}\}.$$

Also in this case one can verify that

$$y_j(y_F - y_{F'} - y_{F''} + y_{F'''}) \in J_{\mathcal{P}}.$$

Let $\mathcal{P}$ be a stair of length $\lambda$ with $\lambda = 4, 6$ or $\lambda \geq 8$ and $\mathcal{C} = \{I_1, \dots, I_\lambda\}$ be its set of cell intervals. If $l_1 = 2 = l_\lambda$ i.e., $\mathcal{P} = \mathcal{S}_\lambda$, then by Proposition 4, $S/I_\mathcal{P}$ is not level. Next we consider the case when $l_1 > 2$ or $l_\lambda > 2$ i.e., $\mathcal{P} = \tilde{\mathcal{S}}_\lambda$.

**Corollary 6** *Let $\mathcal{P} = \tilde{\mathcal{S}}_\lambda$ be a stair of length $\lambda$ with $\lambda = 4, 6$ or $\lambda \geq 8$ and $\mathcal{C} = \{I_1, \dots, I_\lambda\}$ be its set of cell intervals. Let $l_1 > 2$ or $l_\lambda > 2$. Then $S/I_\mathcal{P}$ is not level.*

*Proof* Let $I_1 = \{C_1, \dots, C_{l_1}\}$, $I_i = \{C_{l_1+i-2}, C_{l_1+i-1}\}$ for $2 \leq i \leq \lambda - 1$ and $I_\lambda =$
$\{C_{l_1+\lambda-2}, C_{l_1+\lambda-1}, \dots, C_{l_1+\lambda-3+l_\lambda}\}$. Also let $F_i = \{C_{l_1}, C_{l_1+2}, \dots, C_{l_1+2i}\}$ for $i \geq 1$.
**Claim:** $y_j y_{F_i} = y_{j-1} y_{F_i'}$, where $F_i' = (F_i \setminus \{C_{l_1+2i}\}) \cup \{C_{l_1+2i+1}\}$ for $2 \leq j \leq l_1$.
*Proof of the claim:* Let $2 \leq j \leq l_1$. Then

$$\begin{aligned} y_j y_{F_i} = (y_j y_{l_1}) y_{l_1+2} \cdots y_{l_1+2i} &= y_{j-1}(y_{l_1+1} y_{l_1+2}) \cdots y_{l_1+2i} \\ &= y_{j-1} y_{l_1}(y_{l_1+3} y_{l_1+4}) \cdots y_{l_1+2i} \\ &= \cdots \\ &= y_{j-1} y_{l_1} y_{l_1+2} \cdots (y_{l_1+2i-1} y_{l_1+2i}) \\ &= y_{j-1} y_{l_1} y_{l_1+2} \cdots y_{l_1+2i-2} y_{l_1+2i+1} = y_{j-1} y_{F_i'}. \end{aligned}$$

Let us consider the stair $\mathcal{S}_\lambda$ with cells $\{C_{l_1-1}, C_{l_1}, \dots, C_{l_1+\lambda-2}, C_{l_1+\lambda-1}\}$ of length $\lambda$. First we assume that $\lambda$ is even and $\lambda \geq 4$. Consider $F = \{C_{l_1}, C_{l_1+2}, \dots, C_{l_1+\lambda-2}\}$ and $F' = \{C_{l_1-1}, C_{l_1+2}, C_{l_1+4}, \dots, C_{l_1+\lambda-4}, C_{l_1+\lambda-1}\}$. Then by the proof of Proposition 4, we have that $y_j(y_F - y_{F'}) \in J_\mathcal{P}$ for all $l_1 + 1 \leq j \leq l_1 + \lambda - 3$. For $j = 1$, it is clear that $y_j y_F, y_j y_{F'} \in J_0 \subseteq J_\mathcal{P}$. Let $2 \leq j \leq l_1$. Then by the Claim, we have $y_j y_F = y_{j-1} y_{F''}$, where $F'' = (F \setminus \{C_{l_1+\lambda-2}\}) \cup \{C_{l_1+\lambda-1}\}$. Also, we have

$$y_j y_{F'} = (y_j y_{l_1-1}) y_{l_1+2} \cdots y_{l_1+\lambda-4} y_{l_1+\lambda-1} = y_{j-1} y_{F''}.$$

Therefore, $y_j(y_F - y_{F'}) \in J_\mathcal{P}$ for all $1 \leq j \leq l_1$. Similarly, one can show that $y_j(y_F - y_{F'}) \in J_\mathcal{P}$ for all $l_1 + \lambda - 2 \leq j \leq l_1 + \lambda - 3 + l_\lambda$. Thus we get $y_j(y_F - y_{F'}) \in J_\mathcal{P}$ for all $j$, hence $y_F - y_{F'} \in \mathrm{Soc}(R/J_\mathcal{P})$.

Now assume that $\lambda$ is odd and $\lambda = 2m - 1$ such that $m$ is odd. We consider as in the proof of Proposition 4

$$F = \{C_{l_1}, C_{l_1+2}\} \cup \{C_{l_1+2k-3} : 4 \leq k \leq m\},$$

$$F' = \{C_{l_1-1}, C_{l_1+3}\} \cup \{C_{l_1+2k-3} : 4 \leq k \leq m\},$$

$$F'' = \{C_{l_1-2+2k} : k \in \{1, 2, \dots, m-t-1, m-t+1, \dots, m\}\} \text{ and } F''' = \{C_{l_1-1}\} \cup$$
$$\{C_{l_1-2+2k} : k \in \{2, \dots, m-t-2\}\} \cup \{C_{l_1+2(m-t)-3}\} \cup$$
$$\cup \{C_{l_1-2+2k} : k \in \{m-t+1, \dots, m\}\}.$$

Note that $y_1 y_F, y_1 y_{F'}, y_1 y_{F''}, y_1 y_{F'''} \in J_\mathcal{P}$. Let $2 \leq j \leq l_1$. Then $y_j y_{l_1} y_{l_1+2} = y_{j-1} y_{l_1} y_{l_1+3}$ and $y_j y_{l_1-1} y_{l_1+3} = y_{j-1} y_{l_1} y_{l_1+3}$ which further implies that $y_j(y_F -$

$y_{F'}) \in J_\mathcal{P}$ for $2 \leq j \leq l_1$. Let $F_1'' = \{C_{l_1-2+2k} : k \in \{1,2,\dots,m-t-1\}\} \subseteq F''$. Then by *Claim*, for $2 \leq j \leq l_1$, we have $y_j y_{F_1''} = y_{j-1} y_{F_2''}$, where $F_2'' = (F_1'' \setminus \{C_{l_1+2(m-t)-4}\}) \cup \{C_{l_1+2(m-t)-3}\}$. Also, for $F_1''' = \{C_{l_1-1}\} \cup \{C_{l_1-2+2k} : k \in \{2,\dots,m-t-2\}\} \cup \{C_{l_1+2(m-t)-3}\} \subseteq F'''$, we have $y_j y_{F_1'''} = y_{j-1} y_{F_2''}$. Therefore, $y_j(y_{F''} - y_{F'''}) \in J_\mathcal{P}$ for $1 \leq j \leq l_1$. Similarly, one can show that $y_j(y_F - y_{F'}), y_j(y_{F''} - y_{F'''}) \in J_\mathcal{P}$ for all $l_1+\lambda-2 \leq j \leq l_1+\lambda-3+l_\lambda$. Hence,

$$y_j(y_F - y_{F'} - y_{F''} + y_{F'''}) \in J_\mathcal{P} \quad \forall j.$$

This completes the proof.

**Definition 9** A stair $\mathcal{P}$ of length $\lambda$ with $\lambda = 4, 6$ or $\lambda \geq 8$ is called a *bad stair.*

**Theorem 9** *Let $\mathcal{P}$ be a path polyomino containing a bad stair. Then $S/I_\mathcal{P}$ is not level.*

*Proof* Assume that $\mathcal{P}$ contains a bad stair $\mathcal{S}_\lambda$ or $\tilde{\mathcal{S}}_\lambda$. Here, we show that if $\mathcal{P}$ contains $\tilde{\mathcal{S}}_\lambda$, then $S/I_\mathcal{P}$ is not level. The case when $\mathcal{P}$ contains $\mathcal{S}_\lambda$ is similar. According to Corollary 6, $\tilde{\mathcal{S}}_\lambda$ is not level, hence let $f_\lambda \in R_{\tilde{\mathcal{S}}_\lambda}/J_{\tilde{\mathcal{S}}_\lambda}$ such that $f_\lambda \in \mathrm{Soc}(R_{\tilde{\mathcal{S}}_\lambda}/J_{\tilde{\mathcal{S}}_\lambda})$ and $\deg(f_\lambda) < r(\tilde{\mathcal{S}}_\lambda)$. The stair $\tilde{\mathcal{S}}_\lambda$ is embedded in $\mathcal{P}$ in some intervals $I_{k+1}, I_{k+2}, \dots, I_{k+\lambda}$ with $I_{k+1} = \{C_1^{k+1}, C_2^{k+1}, \dots, C_{r_{k+1}}^{k+1}\}$ where $l(I_{k+1}) = r_{k+1}$. Let $\mathcal{Q}$ be the collection of cells having maximal intervals $\mathcal{C} \setminus \{I_{k+1}, \dots, I_{k+\lambda}\}$, in particular it is the union of two path polyominoes $\mathcal{P}_1$ and $\mathcal{P}_2$. Let $y_{G_i} \in \mathrm{Soc}(R_{\mathcal{P}_i}/J_{\mathcal{P}_i})$ with $\deg y_{G_i} = r(\mathcal{P}_i)$. We now show that

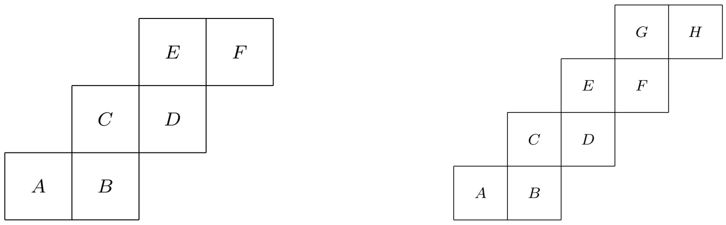
$$y_{G_1} y_{G_2} f_\lambda \in \mathrm{Soc}(R/J_\mathcal{P}).$$

Since $f_\lambda \in \mathrm{Soc}(R_{\tilde{\mathcal{S}}_\lambda}/J_{\tilde{\mathcal{S}}_\lambda})$, then it follows from the proof of Corollary 6 that for all $C_j \in \tilde{\mathcal{S}}_\lambda \setminus \{C_1^{k+1}\}$, we have $y_j f_\lambda \in J_{\tilde{\mathcal{S}}_\lambda} \subseteq J_\mathcal{P}$. Also for similar reason, we can show the claim of Corollary 6 for $y_j$ where $C_j = C_1^{k+1}$. Therefore, we can conclude that $y_j f_\lambda \in J_\mathcal{P}$ for all $C_j \in \tilde{\mathcal{S}}_\lambda$. This implies that $y_j(y_{G_1} y_{G_2} f_\lambda) \in J_\mathcal{P}$ for all $C_j \in \tilde{\mathcal{S}}_\lambda$. Let $C_j \in \mathcal{P}_i$ be any cell. Then $y_j y_{G_i} \in J_{\mathcal{P}_i} \subseteq J_\mathcal{P}$ for all $C_j \in \mathcal{P}_i$ and $i = 1, 2$. Therefore, $y_j(y_{G_1} y_{G_2} f_\lambda) \in J_\mathcal{P}$ for all $C_j \in \mathcal{P}_1 \sqcup \mathcal{P}_2$, and hence $y_{G_1} y_{G_2} f_\lambda \in \mathrm{Soc}(R/J_\mathcal{P})$. Since $\deg(y_{G_1} y_{G_2} f_\lambda) < r(\mathcal{P}_1) + r(\mathcal{P}_2) + r(\tilde{\mathcal{S}}_\lambda) = r(\mathcal{P})$, by Corollary 4, $\mathrm{Soc}(R/J_\mathcal{P})$ has elements of at least two different degrees. Thus, $R/J_\mathcal{P}$ is not level and hence, $S/I_\mathcal{P}$ is not level, too.

**Proposition 5** *Let $\mathcal{P} = \mathcal{S}_\lambda$ be a stair of length $\lambda$. The followings are equivalent:*

1. *$S/I_\mathcal{P}$ is level;*
2. *$\lambda = 2, 3, 5, 7$.*

*Proof* (1) $\Rightarrow$ (2) follows from Proposition 4. Even if (2) $\Rightarrow$ (1) can be showed by direct computation, we want to give a direct proof. In the case $\lambda = 2, 3$, the rook number is 2, hence the assertion follows from Corollary 5. For the case $\lambda = 5, 7$, we will make use of Lemma 6. In fact we have to prove that any $F \in \mathcal{R}_\mathcal{P}$ with $|F| = r(\mathcal{P}) - 1$ satisfies the hypothesis of Lemma 6.




(A) The stair $\mathcal{S}_5$ (B) The stair $\mathcal{S}_7$

Fig. 8: The stairs $\mathcal{S}_5$ and $\mathcal{S}_7$

We refer to the labellings given in Figure 8. If $\lambda = 5$, then $r(\mathcal{P}) = 3$ and the unique facet of cardinality 2 is $\{B, E\}$ and both $B$ and $E$ satisfy the hypothesis of Lemma 6.
If $\lambda = 7$, then $r(\mathcal{P}) = 4$ and the facets of cardinality 3 are

$$\{A, D, G\}, \{B, D, G\}, \{B, E, G\}, \{B, E, H\}.$$

The cells that satisfy the hypothesis of Lemma 6 are respectively $D$,$G$,$B$ and $E$.

**Theorem 10** *Let $\mathcal{P}$ be a path polyomino. The followings are equivalent:*

1. *$S/I_{\mathcal{P}}$ is level;*
2. *$\mathcal{P}$ does not contain bad stairs.*

*Proof* From Theorem 9, we have that (1) $\Rightarrow$ (2). We now prove (2) $\Rightarrow$ (1). If $\mathcal{P}$ does not contain maximal interval of length 2, then by Theorem 8, it is level. Hence assume it contains intervals of length 2. Since $\mathcal{P}$ does not contain bad stair, if $\mathcal{P}$ contains a stair $\mathcal{S}_\lambda$ or $\tilde{\mathcal{S}}_\lambda$ then it must be true that $\lambda \in \{2, 3, 5, 7\}$. It is enough to show that there is no element of degree $< r(\mathcal{P})$ in $\mathrm{Soc}(R/J_{\mathcal{P}})$. Let $f \in \mathrm{Soc}(R/J_{\mathcal{P}})$ be an element of degree $< r(\mathcal{P})$ with $\mathrm{in}(f) = u$. Then by Lemma 5, $u \in \mathrm{in}(\mathrm{Soc}(R/J_{\mathcal{P}})) \subseteq \mathrm{Soc}(R/\,\mathrm{in}(J_{\mathcal{P}}))$ which implies that $u$ can be written as $u = y_F$ for some $F \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ with $|F| < r(\mathcal{P})$. Therefore, $F$ contains a non-single cell $C_k$ in the intersection of two intervals $I_j$ and $I_{j+1}$. If both intervals have a single cell, $C_{k-1}$ and $C_{k+1}$, that is $I_j, I_{j+1}$ is a stair $\tilde{\mathcal{S}}_2$, then we are in the hypotheses of Lemma 6, $(F\setminus\{C_k\})\cup\{C_{k-1}, C_{k+1}\} \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ which is a contradiction by Lemma 6. Moreover, if one between $I_j$ or $I_{j+1}$ has no single cell, say $I_{j+1}$, then $I_{j+1}$ belongs to a stair $\mathcal{S}_\lambda$ (resp. $\tilde{\mathcal{S}}_\lambda$) with $\lambda \in \{3, 5, 7\}$. The case $\lambda = 3$ can be eliminated by the following observation: since $l_{j+1} = 2$, then $l_j, l_{j+2} > 2$ and in particular, the cell $C_{k-1}$ in $I_j$ is single, and one can take $(F\setminus\{C_k\})\cup\{C_{k-1}, C_{k+1}\} \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$. Hence, we are left with the case $\lambda \in \{5, 7\}$. Observe that $F$ contains a facet $F' \in \mathcal{F}(\mathcal{R}_{\mathcal{S}_\lambda})$ with $C_k \in F'$ and $|F'| < r(\mathcal{S}_\lambda)$. Then $F'$ is one of the form given in the proof of Proposition 5. Therefore, $F'$ contains a $C_m$ such that $(F' \setminus \{C_m\}) \cup \{C_{m-1}, C_{m+1}\} \in \mathcal{F}(\mathcal{R}_{\mathcal{S}_\lambda})$. Hence, $(F \setminus \{C_m\}) \cup \{C_{m-1}, C_{m+1}\} \in \mathcal{F}(\mathcal{R}_{\mathcal{P}})$ which is a contradiction by Lemma 6. This completes the proof.

In Figure 9, we show an example of polyomino containing a bad stair $\tilde{\mathcal{S}}_4$, given by the cells $C_3, \dots, C_9$, and hence not level.

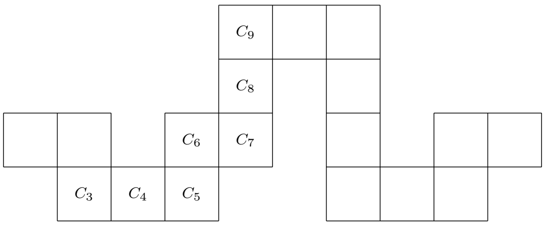



Fig. 9: A path polyomino containing a bad stair

## 5 Levelness and Pseudo-Gorensteinnes of simple thin polyominoes

In [14], we developed different routines to generate polyominoes and test their primality. After a slight modification of the code provided in that paper, that is possible to download from [21], we generated all simple thin polyominoes, classifying them, by using Macaulay2 (see [8]), with respect to the following properties:

(G) Gorenstein;
(PG) Pseudo-Gorenstein (not Gorenstein);
(L) Level (not Gorenstein);
(N) None of the above.

In Figure 13, we display all the non-path simple thin polyominoes of rank 6. We observe that they are all level, but not Gorenstein.

| **Rank** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
|---|---|---|---|---|---|---|---|
| Gorenstein | 0 | 3 | 0 | 10 | 0 | 47 | 0 |
| Level | 4 | 7 | 26 | 65 | 230 | 684 | 2383 |
| pseudo-Gorenstein | 0 | 1 | 0 | 5 | 0 | 36 | 0 |
| None of the above | 0 | 0 | 1 | 2 | 20 | 48 | 302 |

Table 1: The partition of all simple thin polyominoes of rank less than or equal to 10

In the website [21], it is possible to download all the simple thin polyominoes, with respect to the previous partition, having rank in the set $\{4, \dots, 10\}$.

*Remark 5* We observe that the polyominoes of rank 1, 2, and 3 are paths and are studied in the previous sections. In particular, the single cell is Gorenstein, the domino (the polyomino with 2 cells) is level, and there are 2 paths with rank 3: one is level and the other is Gorenstein. By the Table 1 we observe that the pseudo-Gorenstein simple thin polyominoes have odd rank in the interval $\leq 10$.

Inspired by Remark 5 and Corollary 1 we obtain the following

**Theorem 11** *Let $\mathcal{P}$ be a simple thin pseudo-Gorenstein polyomino. Then* $\operatorname{rk}\mathcal{P} = 2r(\mathcal{P}) - 1$.

*Proof* We use induction on $\operatorname{rk}\mathcal{P}$. The cases $\operatorname{rk}\mathcal{P} = 1, 2, 3, 4$ are in Remark 5 and Table 1. Suppose by induction hypothesis that for a fixed $k$ and for all $\mathcal{P}$ such that $1 \leq \operatorname{rk}\mathcal{P} \leq k$ the equation holds. Now focus on the case $\operatorname{rk}\mathcal{P} = k+1$. We recall that any simple thin polyomino has an interval, say $I$, that is called either a tail or an endcut (see Definition 3.4 of [20]): in Figure 10 the endcut (resp. tail) is the interval $I$ with cells $C$ and $D$. By removing $I$ from $\mathcal{P}$ we obtain the polyomino $\mathcal{P}'$, in both cases.

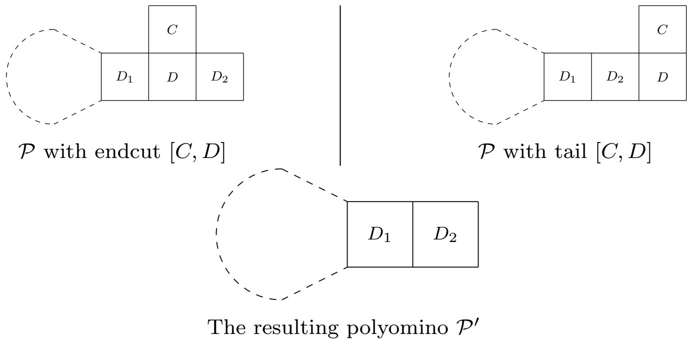



Fig. 10

Now, suppose $\mathcal{P}$ is pseudo-Gorenstein. If $I$ has more than one single cell, then $\mathcal{P}$ is not pseudo-Gorenstein, hence the length of $I$ is 2, as in Figure 10. Let $F$ be the unique configuration of non-attacking rooks of maximum cardinality (see Lemma 1), we observe that $C \in F$. Moreover, call $\mathcal{P}'$ the polyomino $\mathcal{P} \setminus I$ (see again Figure 10). We observe that $\mathcal{P}'$ is pseudo-Gorenstein, too. In fact, if we have two distinct configurations of non-attacking rooks of maximum cardinality in $\mathcal{P}'$, say $F_1$ and $F_2$, then $F_1 \cup \{C\}$ and $F_2 \cup \{C\}$ are distinct configurations of non-attacking rooks of maximum cardinality in $\mathcal{P}$, leading to a contradiction. Hence, the unique configuration of non-attacking rooks of maximum cardinality in $\mathcal{P}'$ is $F' = F \setminus \{C\}$.

Now, suppose $\operatorname{rk}\mathcal{P}$ is even, then $\operatorname{rk}\mathcal{P}' = \operatorname{rk}\mathcal{P} - 2$ is even, too. By induction hypothesis $\mathcal{P}'$ is not pseudo-Gorenstein. Hence $\mathcal{P}$ is not pseudo-Gorenstein.

Let $\operatorname{rk}\mathcal{P} = k+1$ be odd. Then $\mathcal{P}'$ is pseudo-Gorenstein, and it has a unique facet $F$ of maximum cardinality. Moreover, its cardinality is $k/2$. Then, there is a unique facet of maximum cardinality $k/2 + 1$ of $\mathcal{P}$, that is $F$ with the single cell of $I$.

As the rook number of a simple thin polyomino $\mathcal{P}$ coincides with the regularity of $S/I_{\mathcal{P}}$, we obtain the following

**Corollary 7** *Let $\mathcal{P}$ be a simple thin pseudo-Gorenstein polyomino. Then*

$$\operatorname{reg}(S/I_{\mathcal{P}}) = \frac{\operatorname{rk}\mathcal{P}+1}{2}.$$

In Figure 11, we show an example of pseudo-Gorenstein simple thin polyomino. More examples can be found on the webiste [21]

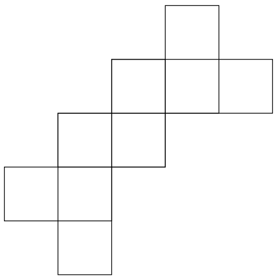


Fig. 11: A pseudo-Gorenstein simple thin polyomino

Motivated by the observation that all the path polyominoes that satisfy Theorem 8 have at least a single cell in any interval, and by computational evidence (e.g. all of the polyominoes in Figure 13 but (1) and (6)), the following conjecture naturally arises.

*Conjecture 1* Let $\mathcal{P}$ be a simple thin polyomino such that any maximal interval has a single cell. Then $S/I_{\mathcal{P}}$ is level.

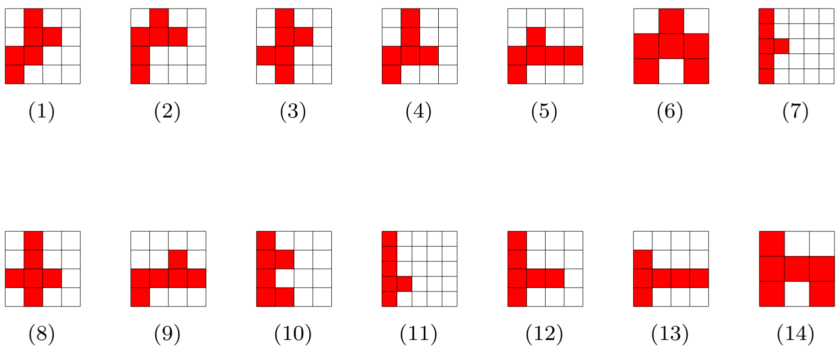



Fig. 13: Simple Thin Polyominoes of Rank 6, that are not paths

In conclusion, it is of interest the following

*Question 1* Is it possible to generalize the concept of (bad) stair to characterize level or pseudo-Gorenstein simple thin polyominoes?

**Data Availability Statement** The datasets generated during the current study are available in the first author's repository, [21].

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