

2-Dimensional vertex decomposable circulant graphs

by

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Abstract

Let G be the circulant graph $C_n(S)$ with $S \subseteq \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ and let Δ be its independence complex. We describe the well-covered circulant graphs with 2-dimensional Δ , and construct an infinite family of vertex-decomposable circulant graphs within this family. Moreover, we show that if $C_n(S)$ has a 2-dimensional vertex decomposable Δ , then it has a level Stanley-Reisner ring.

Key Words: Circulant graphs, Cohen-Macaulay, vertex decomposability.

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Introduction

Let $n \in \mathbb{N}$ and $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $G := C_n(S)$ is a graph with vertex set $\mathbb{Z}_n = \{0, \dots, n-1\}$ and edge set $E(G) := \{\{i, j\} \mid |j - i|_n \in S\}$ where $|k|_n = \min\{|k|, n - |k|\}$.

Let $R = K[x_0, \dots, x_{n-1}]$ be the polynomial ring on n variables over a field K . The *edge ideal* of G , denoted by $I(G)$, is the ideal of R generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Edge ideals of graphs have been introduced by Villarreal [20] in 1990, where he studied the Cohen-Macaulay property of such ideals. Many authors have focused their attention on such ideals (e.g. [10], [6]). A known fact about Cohen-Macaulay edge ideals is that they are *well-covered*, that is all the maximal independent sets of G have the same cardinality. Despite the nice structure circulant graphs have, it has been proved that their clique number is hard to compute (see [5]), and hence so is the Krull dimension of $R/I(G)$.

In particular, some well-covered circulant graphs have been studied in [2], [3], [19], [8] and [15]. In [19] and [8] the authors studied well-covered circulant graphs that are Cohen-Macaulay. The most interesting families are power cycle ones and their complements. Within these families, the Cohen-Macaulay ones have Krull dimension 2. Moreover, the infinite family of well-covered power cycles of Krull dimension 3 has elements that are all Buchsbaum ([19]). In addition, in [8, Table 1] the authors studied all the circulant graphs within 16 vertices, by using symbolic computation. Among these, the Cohen-Macaulay ones with Krull dimension 3 have $R/I(G)$ that is the tensor product of Cohen-Macaulay rings of Krull dimension 1 (e.g. $C_6(3)$, $C_9(3)$, etc.). We observe that the first non-trivial Cohen-Macaulay circulant graph with Krull dimension 3 is the Paley graph $C_{17}(\overline{1}, 2, 4, 8)$ (see Example 1). In particular we verified, through a `Macaulay2` computation, that the independence complex of $C_{17}(\overline{1}, 2, 4, 8)$ is also vertex decomposable (see [12]). Hence a natural question arises: “Is it possible to find an infinite family of circulant graphs of Krull dimension 3 that are Cohen-Macaulay?”

The idea is to find good properties on n and S to find such a family. In fact we will prove the following result.

Theorem 1. *Let $G = C_n(\overline{1, 2, 4, \dots, 2^m, 2^m - 1})$ with $m \geq 3$ and $n = 3 \cdot 2^m$ and let Δ be its independence complex. Then Δ is a 2-dimensional vertex decomposable simplicial complex with respect to the following sequence of vertices*

$$[1, 2, \dots, \widehat{2^m}, 2^m + 1, \dots, \widehat{2^{m+1}}, \dots, n - 1].$$

We recall that the vertex decomposability implies the Cohen-Macaulayness.

In Section 2 we give a characterization of pure 2-dimensional independence complexes of circulants (Proposition 1) and explicit formulas for the f -vector and h -vector (Proposition 2 and Proposition 3). In Section 3, we give the proof of Theorem 1. Moreover, we present an example, where $n = 3 \cdot 2^3$, to clarify the steps of the proof of Theorem 1 (Example 2). Furthermore in Section 4, we prove that any 2-dimensional vertex decomposable independence complex of circulant graphs has Stanley-Reisner ring that is a level algebra (Theorem 3). It is known that the Hilbert function of level algebras has nice properties (see [9]). Moreover when one talks about level algebras, the question about which ones are also Gorenstein algebras naturally arises (see [4], [9]). In this regard, in Proposition 4 we prove that among the level algebras of Theorem 3, the only Gorenstein algebra is $R/I(G)$ where $G = C_6(3)$.

1 Preliminaries and the Paley example

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article.

Set $V = \{x_1, \dots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that: 1) $\{x_i\} \in \Delta$ for all $x_i \in V$; 2) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ . A maximal face of Δ with respect to inclusion is called a *facet* of Δ .

The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of Δ is the maximum of the dimensions of all facets. Moreover, if all the facets of Δ have the same dimension, then we say that Δ is *pure*. Let $d - 1$ the dimension of Δ and let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1} = 1$. Then the f -vector of Δ is the $(d + 1)$ -tuple $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$. The h -vector of Δ is $h(\Delta) = (h_0, h_1, \dots, h_d)$ with

$$h_k = \sum_{i=0}^k (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}. \tag{1.1}$$

The sum

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i$$

is called the *reduced Euler characteristic* of Δ and $h_d = (-1)^{d-1} \tilde{\chi}(\Delta)$. For any $F \in \Delta$ we define $\text{link}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset \text{ and } F \cup G \in \Delta\}$, $\text{del}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset\}$. We define the *chain complex* as follows:

$$\mathcal{C} : 0 \rightarrow K^{f_{d-1}} \xrightarrow{\partial_{d-1}} K^{f_{d-2}} \xrightarrow{\partial_{d-2}} \dots \xrightarrow{\partial_0} K \rightarrow 0$$

and by definition the i -th *reduced homology group* $\tilde{H}_i(\Delta; K)$ is

$$\tilde{H}_i(\Delta; K) = \ker(\partial_i) / \text{im}(\partial_{i+1}).$$

We set $b_i = \dim \tilde{H}_i(\Delta; K)$ and we point out that

$$b_0 = c - 1 \tag{1.2}$$

where c is the number of distinct components of Δ . As in [11, Chapter 7], the reduced Euler characteristic $\tilde{\chi}(\Delta)$ can be seen as

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i b_i = (-1)^{d-1} h_d. \tag{1.3}$$

Given any simplicial complex Δ on V , we can associate a monomial ideal I_Δ in the polynomial ring R as follows:

$$I_\Delta = (\{x_{j_1} x_{j_2} \cdots x_{j_r} : \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \notin \Delta\}).$$

R/I_Δ is called the *Stanley-Reisner ring* (see [17]) and its Krull dimension is d . If G is a graph we denote the *independence complex* of G by

$$\Delta(G) = \{A \subset V(G) : A \text{ is an independent set of } G\}.$$

The *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G .

Let $T = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ and G be a circulant graph on $S \subseteq T$. We observe that \bar{G} is a circulant graph on $\bar{S} = T \setminus S$ and the clique complex of \bar{G} is the independence complex of G , $\Delta(G)$. So from now on we will take Δ as the clique complex of the graph $\bar{G} = C_n(\bar{S})$.

Let Δ be a pure independence complex of a graph G . We say that Δ is *vertex decomposable* if one of the following conditions hold: (1) $n = 0$ and $\Delta = \{\emptyset\}$; (2) Δ has a unique maximal facet $\{x_0, \dots, x_{n-1}\}$; (3) There exists $x \in V(G)$ such that both $\text{link}_\Delta(x)$ and $\text{del}_\Delta(x)$ are vertex decomposable and the facets of $\text{del}_\Delta(x)$ are also facets in Δ .

We say that Δ is *Cohen-Macaulay* if for any $F \in \Delta$ we have that $\dim_K \tilde{H}_i(\text{link}_\Delta(F), K) = 0$ for any $i < \dim \text{link}_\Delta(F)$. In particular, Δ is Cohen-Macaulay if and only if R/I_Δ is a Cohen-Macaulay ring (see [4]).

We say that Δ is *Buchsbaum* if $\forall \{x\} \in \Delta$ we have that $\text{link}_\Delta(x)$ is Cohen-Macaulay. In particular, Buchsbaum simplicial complexes were studied in [18, 14]. It is well known that

$$\begin{aligned} \Delta \text{ Vertex Decomposable} &\Rightarrow \\ &\Rightarrow \Delta \text{ Cohen-Macaulay} \Rightarrow \Delta \text{ Buchsbaum} \Rightarrow \Delta \text{ Pure.} \end{aligned}$$

Remark 1. Let Δ be a 0-dimensional simplicial complex on n vertices. Then Δ is vertex decomposable.

Lemma 1. Let Δ be a 1-dimensional simplicial complex on n vertices. Then the following are equivalent

- (i) Δ is vertex decomposable;
- (ii) Δ is connected.

Let \mathbb{F} be the minimal free resolution of $R/I(G)$. Then

$$\mathbb{F} : 0 \rightarrow F_p \rightarrow F_{p-1} \rightarrow \dots \rightarrow F_0 \rightarrow R/I(G) \rightarrow 0$$

where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$. The $\beta_{i,j}$ are called the *Betti numbers* of \mathbb{F} . For any i , $\beta_i = \sum_j \beta_{i,j}$ is called the i -th *total Betti number*. The *Castelnuovo-Mumford regularity* of $R/I(G)$, denoted by $\text{reg } R/I(G)$ is defined as

$$\text{reg } R/I(G) = \max\{j - i : \beta_{i,j} \neq 0\}.$$

Let $\sigma \subseteq V = \{x_0, \dots, x_{n-1}\}$. We define the *restriction* of the simplicial complex Δ to σ as

$$\Delta|_\sigma = \{F \in \Delta \mid F \subseteq \sigma\}.$$

Theorem 2 (Hochster’s formula, [13]). *The non-zero Betti numbers of R/I_Δ lie in the squarefree degree j , and we have*

$$\beta_{i,j}(R/I_\Delta) = \sum_{|\sigma|=j, \sigma \subseteq V} \dim_K \tilde{H}_{j-i-1}(\Delta|_\sigma; K).$$

If $R/I(G)$ is Cohen-Macaulay, then the last total Betti number β_p is the *Cohen-Macaulay type* of $R/I(G)$. Moreover, if the Cohen-Macaulay type is $\beta_{p,p+\text{reg } R/I(G)}$, then $R/I(G)$ is called a *level algebra*. When the Cohen-Macaulay type is equal to 1, we say that $R/I(G)$ is a *Gorenstein algebra*.

Lemma 2. *Let G be a Cohen-Macaulay graph with independence complex Δ such that $\tilde{\chi}(\Delta) \neq 0$ and Cohen Macaulay type s . Then $R/I(G)$ is a level algebra if and only if $s = |\tilde{\chi}(\Delta)|$.*

Proof. We have to compute $\beta_{p,p+r}$, where $p = \text{pd } R/I(G)$ and $r = \text{reg } R/I(G)$. Let d be the Krull dimension of $R/I(G)$. Since $R/I(G)$ is Cohen-Macaulay and from Auslander-Buchsbaum formula, we have that $\text{pd } R/I(G) = n - d$. Moreover, since $\tilde{\chi}(\Delta) \neq 0$ from [16, Remark 1.2] and [7, Corollary 4.8], $\text{reg } R/I(G) = \text{depth } R/I(G) = d$. Hence,

$$\beta_{n-d,n} \stackrel{(*)}{=} \dim \tilde{H}_{d-1}(\Delta; K) \stackrel{(**)}{=} |\tilde{\chi}(\Delta)|,$$

where $(*)$ follows from Theorem 2 and $(**)$ from the fact that Δ is Cohen-Macaulay and from Equation (1.3). Then the assertion follows. □

We now give an example of a circulant graph whose independence complex is vertex decomposable.

Example 1. *Let us consider the circulant graph $G = C_{17}(\overline{1, 2, 4, 8})$, that is the the Paley 17 (see [1]). In Figure 1 we represent $\text{link}_\Delta(x)$ for an $x \in V(G)$.*

This graph is the first example of Cohen-Macaulay circulant graph whose Stanley-Reisner ring has Krull dimension 3 and it is not a tensor product of rings having Krull dimension 1. Moreover, it is a level algebra.

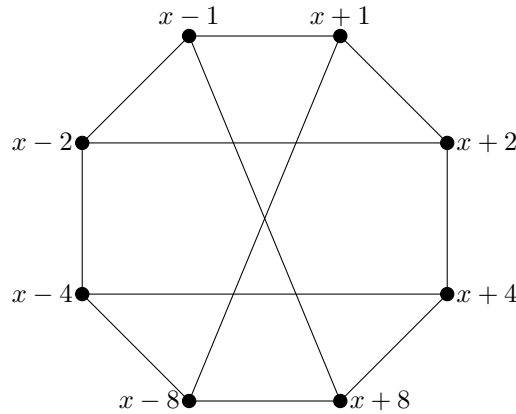


Figure 1: $\text{link}_\Delta(x)$ for any $x \in V(C_{17}(\overline{1, 2, 4, 8}))$

2 2-dimensional well-covered independence complexes

We start by providing a description of well-covered graphs of Krull dimension 3 in terms of the elements in S .

Proposition 1. *Let $G = C_n(S)$ be a non-complete circulant graph. Then Δ is a pure simplicial complex of $\dim \Delta = 2$ if and only if for any $a \in \overline{S}$ the following conditions hold*

1. *There exists $|b|_n \in \overline{S}$ such that $|b - a|_n \in \overline{S}$;*
2. *For any $|b|_n, |c|_n \in \overline{S}$ with $b \neq c$ and such that $|c - a|_n, |b - a|_n \in \overline{S}$ then $|b - c|_n \notin \overline{S}$.*

Proof. \Rightarrow) Let a be such that $|a|_n \in \overline{S}$. Suppose by contraposition that at least one of the conditions (1) or (2) does not hold. Firstly, let us suppose that there does not exist $b \in V(G)$ such that $|b|_n, |b - a|_n \in \overline{S}$, then the edge $\{0, a\}$ is a facet of dimension 1 of Δ . It contradicts the assumption. Secondly, if there exist $b, c \in V(G)$ such that $|b|_n, |c|_n, |c - a|_n, |b - a|_n \in \overline{S}$ and $|b - c|_n \in \overline{S}$ then $\{0, a, b, c\}$ is a facet of dimension 3 of Δ . It contradicts the assumption. \Leftarrow) We start by proving that Δ is 2-dimensional. From (1) it follows that $\{0, a, b\} \in \Delta$, namely $\dim \Delta \geq 2$. Now we prove $\dim \Delta \leq 2$. By contraposition let $\dim \Delta > 2$, then there exists a facet of dimension 3, namely $\{0, a, b, c\}$ so that

$$|a|_n, |b|_n, |c|_n, |c - a|_n, |b - a|_n, |b - c|_n \in \overline{S}$$

that contradicts (2).

Now we prove that Δ is pure. By contraposition, assume Δ is 2-dimensional but not pure. Since G is not complete, \overline{S} is not empty, namely there are no isolated vertices. Then there exists $a \in V(G)$ such that $\{0, a\}$ is a facet of Δ , and in particular $|a|_n \in \overline{S}$. It contradicts the assumption (1). \square

Now we prove some properties on the f -vector and the h -vector of 2-dimensional independence complexes of circulants.

Proposition 2. . Let $G = C_n(S)$ be such that $\dim \Delta = 2$ and

$$\mathcal{F}_0 = \left\{ \{0, a, b\} \subset V : |a|_n, |b|_n, |b-a|_n \in \bar{S} \right\},$$

the set of the 2-dimensional facets of Δ containing the vertex 0 and let

$$\mathcal{F}_0 = \mathcal{T} \sqcup \mathcal{T}_e$$

where

$$\mathcal{T}_e = \left\{ \{0, a, b\} \subset V : |a|_n = |b|_n = |b-a|_n \in \bar{S} \right\}.$$

Then $|\mathcal{T}| = 3t$, for some $t \in \mathbb{N}$ and

$$|\mathcal{T}_e| = \begin{cases} 1 & \text{if } n = 3k \text{ with } k \in \bar{S} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For any $F = \{0, a, b\}$ in \mathcal{T} by shifting the elements of F by a and b , we obtain the sets $F(-a) = \{-a, 0, b-a\}$, $F(-b) = \{-b, a-b, 0\}$ that are distinct and belong to \mathcal{T} . These are the only shifts sending $F \in \mathcal{F}_0$ in $F' \in \mathcal{F}_0$. Hence

$$3 \mid |\mathcal{T}|.$$

By similar argument if $F \in \mathcal{T}_e$ through the shifts we obtain F itself. \square

Remark 2. We highlight that there is no circulant $C_{2k}(S)$ such that $k \in \bar{S}$ with pure and 2-dimensional Δ . In fact by contraposition let us assume such a G exists. Since Δ is pure, there exists at least an a such that $\{0, a, k\}$ is a 2-face of Δ , with $|k-a|_n \in \bar{S}$. If we set $a = a$, $b = k$ and $c = a+k$ we have that $|b|_n, |c|_n = |k-a|_n, |c-a|_n = |k|_n, |b-a|_n, |b-c|_n \in \bar{S}$ that contradicts (2) of Proposition 1.

Proposition 3. Let $G = C_n(S)$ be such that Δ is a pure simplicial complex of dimension 2. Then $h(\Delta) = (1, n-3, n(s-2)+3, h_3)$ with

$$h_3 = \begin{cases} -1 + n(t-s+1) + k & \text{if } n = 3k \text{ with } k \in \bar{S} \\ -1 + n(t-s+1) & \text{otherwise} \end{cases} \quad (2.1)$$

where $s = |\bar{S}|$ and $t = \frac{1}{3}|\mathcal{T}|$.

Proof. By plugging $k = 1, 2$ and $d = 3$ in formula (1.1), we obtain

$$h_1 = f_0 - 3; \quad h_2 = f_1 - 2n + 3.$$

Since from Remark 2 if n is even $\frac{n}{2} \notin \bar{S}$, we have $f_1 = ns$. Moreover,

$$h_3 = \tilde{\chi}(\Delta) = -1 + f_0 - f_1 + f_2 = -1 + n - ns + f_2.$$

From [15, Lemma 1] we have $f_2 = \frac{n \cdot |\mathcal{F}_0|}{3}$, where $|\mathcal{F}_0|$ is the number of 2-dimensional facets of Δ containing the vertex 0. By notation of Proposition 2, we set

$$t := \frac{1}{3}|\mathcal{T}|.$$

If $n \neq 3k$ or $n = 3k$ with $k \notin \bar{S}$ we have $|\mathcal{T}_e| = 0$ and $|\mathcal{F}_0| = 3t$. In the case $n = 3k$ and $k \in \bar{S}$ we have $|\mathcal{F}_0| = 3t + 1$, that yields $f_2 = \frac{n(3t+1)}{3} = nt + k$. Hence (2.1) follows. \square

3 Proof of Theorem 1

The aim of this section is to prove Theorem 1. We first prove that Δ is pure and 2-dimensional, computing its f -vector.

Proposition 4. *Let $G = C_n(\overline{1, 2, 4, \dots, 2^m, 2^m - 1})$, $m \geq 3$ and $n = 3 \cdot 2^m$. Then Δ is a pure 2-dimensional simplicial complex with f -vector*

$$(1, n, n(m + 2), n(m + 2) + 2^m).$$

Proof. We prove that Δ is a pure 2-dimensional simplicial complex, by using Proposition 1. For this aim, we describe the 2-faces of Δ containing the vertex 0. In the notation of Proposition 2, $\{0, 2^m, 2^{m+1}\} \in \mathcal{T}_e$ and \mathcal{T} is formed by the elements $F \in \{\{0, a, b\} : a, b \in \bar{S}\}$ that are

$$\{0, 2^i, 2^{i+1}\}_{i=0, \dots, m-1}, \{0, 1, 2^m\}, \{0, 2^m - 1, 2^m\}, \tag{3.1}$$

and their shifts $F(-a) = \{-a, 0, b - a\}$, $F(-b) = \{-b, a - b, 0\}$. Therefore for any $a \in \bar{S}$, $\{0, a\}$ is not a facet of Δ , condition (1) of Proposition 1.

To verify condition (2), we claim that there are no faces $\{0, a, b, c\} \in \Delta$. To prove this claim we distinguish two cases:

(C1) $a, b, c \in \bar{S}$;

(C2) $a, b \in \bar{S}$, with $a < b$ and $c \in \{-s : s \in \bar{S}\}$.

By symmetry the other cases follow.

(C1) We need to verify that for all $\{0, a, b\}$ and $\{0, a, c\}$ in (3.1) we have $|b - c| \notin \bar{S}$. For any $i \in \{1, 2, \dots, m - 2\}$ we have $|2^{i+2} - 2^i|_n = 2^{i+2} - 2^i \notin \bar{S}$, then $\{0, 2^i, 2^{i+1}, 2^{i+2}\}$ is not a 3-face of Δ . Furthermore, $|2^m - 2| = 2^m - 2 \notin \bar{S}$ because $m \geq 3$, that is $\{0, 1, 2^m - 1, 2^m\}$, $\{0, 1, 2, 2^m\}$ are not 3-faces of Δ . By similar arguments, the remaining cases follow.

(C2) The strategy is the following. We consider the vertices that are adjacent to both 0 and a , and prove that within this set each pairs of candidates satisfying (C2) are not in Δ .

Let $a = 1$. We observe that the vertices adjacent to 0 and 1 are $\{2, -1, 2^m, 1 - 2^m\}$. The candidates $\{b, c\}$ are

$$\{2, -1\}, \{2, -2^m + 1\}, \{2^m, -1\}, \{2^m, -2^m + 1\}.$$

It is straightforward to see that the above pairs are not in Δ .

Let $a = 2^i$ with $1 \leq i \leq m - 1$. Then the only candidate $\{b, c\}$ is $\{2^{i+1}, -2^i\}$. The latter is not in Δ .

Let $a = 2^m - 1$, then $b = 2^m$ and the only candidate for $c \in \{-s : s \in \bar{S}\}$ is -1 . But $\{2^m, -1\}$ is not in Δ .

Hence we have Δ is pure and 2-dimensional.

For what matters the f -vector of Δ , we have that $f_1 = ns$ where $s = |\bar{S}|$, hence $s = m + 2$. According to Proposition 2 and [15, Lemma 1] we have $f_2 = \frac{n \cdot |\mathcal{F}_0|}{3}$. The elements of \mathcal{F}_0 are the 2-faces F in (3.1) and the shifted ones plus the one of \mathcal{T}_e . Hence, they are $3(m + 2) + 1$, and $f_2 = \frac{3n(m+2)+n}{3} = n(m + 2) + 2^m$. \square

We present a characterization of vertex decomposability for 2-dimensional simplicial complexes useful for our aim.

Lemma 3. *Let Δ be a 2-dimensional pure connected simplicial complex on n vertices, let $\mathcal{M} = [v_1, v_2, \dots, v_{n-3}]$ be a sequence of vertices of Δ , and for $i = 1, 2, \dots, n-3$ let*

$$\Delta_{i-1} = \begin{cases} \Delta & \text{if } i = 1 \\ \text{del}_{\Delta_{i-2}}(v_{i-1}) & \text{otherwise.} \end{cases}$$

Then the following are equivalent:

- (i) Δ is vertex decomposable with respect to \mathcal{M} ;
- (ii) \mathcal{M} satisfies the following properties:
 1. For any $i = 1, 2, \dots, n-3$, $\text{link}_{\Delta_{i-1}}(v_i)$ is a connected 1-dimensional simplicial complex;
 2. Δ_{n-3} is the simplex on 3 vertices.

Proof. (i) \Rightarrow (ii). By contraposition, we assume that one of the following is true:

- (1)' There exists a $k \in \{1, 2, \dots, n-3\}$ such that $\text{link}_{\Delta_{k-1}}(v_k)$ is disconnected or 0-dimensional;
- (2)' Δ_{n-3} is not the simplex on 3 vertices.

If (1)' and there exists a $k \in \{1, 2, \dots, n-3\}$ such that $\text{link}_{\Delta_{k-1}}(v_k)$ is a disconnected and 1-dimensional, then $\text{link}_{\Delta_{k-1}}(v_k)$ is not vertex decomposable according to Lemma 1, hence Δ is not vertex decomposable. If it is 0-dimensional, then there exists an isolated vertex b in $\text{link}_{\Delta_{k-1}}(v_k)$, that is $\{v_k, b\} \in \mathcal{F}(\Delta_{k-1})$ and the facets of Δ_{k-1} are not facets of Δ , that contradicts the assumption of vertex decomposability. If (2)', then $\dim \Delta_{n-3} < 2$ and so the facets of Δ_{n-3} are not facets in Δ .

(ii) \Rightarrow (i). We claim the sequence \mathcal{M} is a sequence that is a vertex decomposition of Δ . From Lemma 1 and the property (ii).(1) we obtain that $\text{link}_{\Delta_{i-1}}(v_i)$ for $i = 1, 2, \dots, n-3$ are vertex decomposable. Hence to prove that Δ is vertex decomposable we are left with proving the following

Claim: For any $i = 1, \dots, n-3$ the facets of Δ_i are facets of Δ .

Let us assume condition (ii).(2) and that there exist a $j \in \{1, 2, \dots, n-3\}$ and $\{a, b\}$ is a facet of Δ_j . Then at least one between a , and b leaves in \mathcal{M} . In fact, if both a, b do not live in \mathcal{M} , then $\{a, b\}$ will be an edge of Δ_{n-3} , that we recall is a simplex on 3 vertices. That is impossible. So let us assume that there exists a $k > i$ such that $a = v_k$ and $\{v_k, b\} \in \mathcal{F}(\Delta_j)$. It implies that $\text{link}_{\Delta_{k-1}}(v_k)$ contains b as isolated vertex, that contradicts the property (ii).(1). Hence the claim follows. \square

Now we prove the main theorem.

Proof of Theorem 1. From Proposition 4, Δ is pure and 2-dimensional. To prove the vertex decomposability of Δ , it is useful to define the following edge sets

$$E(\mathcal{H}_v^l) = \left\{ \{v - 2^i, v - 2^{i+1}\}_{i=l, l+1, \dots, m-1}, \{v - 2^m, v - 2^m - 1\} \right\},$$

$$E(\mathcal{P}_v^l) = \left\{ \{v + 2^i, v + 2^{i+1}\}_{i=0, 1, \dots, l-1} \right\}$$

that are the edges of two paths,

$$E(\mathcal{G}_v) = \left\{ \{v + 2^i, v + 2^{i+1}\}_{i=0, 1, \dots, m-1} \right\} \cup \left\{ \{v + 2^m, v + 2^m - 1\} \right\},$$

$$E(\mathcal{L}_v) = \left\{ \{v - 2^i, v - 2^{i+1}\}_{i=0, 1, \dots, m-1} \right\} \cup \left\{ \{v - 2^m, v - 2^m + 1\} \right\},$$

that are the edges of two cycles with an extra edge and

$$E(\mathcal{B}_v^l) = \left\{ \{v - 2^i, v + 2^i\}_{i=l, l+1, \dots, m} \right\}$$

that are disjoint edges connecting $E(\mathcal{L}_v)$ and $E(\mathcal{G}_v)$. We will prove that Δ is vertex decomposable by using Lemma 3, that is we want to find a sequence of vertices v_1, v_2, \dots, v_{n-3} satisfying (ii).(1) and (ii).(2). We claim that such a sequence is

$$[1, 2, \dots, \widehat{2}^m, 2^m + 1, \dots, \widehat{2}^{m+1}, \dots, n - 1].$$

Let us consider the vertices v in $1, 2, \dots, 2^m - 1$.

For $v = 1$ and $\Delta_0 = \Delta$, $\text{link}_\Delta(1)$ is vertex decomposable. In fact, for any $v \in V(G)$, $\mathcal{F}(\text{link}_\Delta(v))$ is, by abuse of notation,

$$E(\mathcal{L}_v) \cup E(\mathcal{G}_v) \cup E(\mathcal{B}_v^0) \cup \left\{ \{v - 1, v + 2^m - 1\}, \{v + 1, v - 2^m + 1\} \right\}$$

that is 1-dimensional and connected (see Figure 2).

We describe the first steps $v = 2, 3, 4$ before giving the general set (3.2) for $\mathcal{F}(\text{link}_{\Delta_{v-1}}(v))$ with v in the interval $[2, 2^m - 1]$.

For $v = 2$, we have that the vertex $1 = v - 1$ is not in $\text{link}_{\Delta_1}(v)$, hence $\mathcal{F}(\text{link}_{\Delta_1}(v))$ is equal to

$$E(\mathcal{H}_v^1) \cup E(\mathcal{B}_v^1) \cup \left\{ \{v - 2^m + 1, v + 1\} \right\} \cup E(\mathcal{G}_v),$$

that is 1-dimensional and connected (see Figure 3). From now on, we omit the last observation that will be clear by the descriptions of the links.

For $v = 3$, we have that $2 = v - 1$, $1 = v - 2 \notin V(\Delta_2)$, hence the edges $\{v - 2, v + 4\}$ and $\{v - 2, v + 2\}$ are not in Δ_2 (see Figure 4), and $\mathcal{F}(\text{link}_{\Delta_2}(v))$ is

$$E(\mathcal{H}_v^2) \cup E(\mathcal{B}_v^2) \cup \left\{ \{v - 2^m + 1, v + 1\} \right\} \cup E(\mathcal{G}_v).$$

For $v = 4$, the same facts of the case $v = 3$ hold. Hence, $\text{link}_{\Delta_3}(v)$ is isomorphic to $\text{link}_{\Delta_2}(3)$. To get the general set we observe that for $2 \leq v \leq 2^m - 1$, we have two cases: if $v - 1 \in \bar{S}$ we loose the edge $\{v - 2^l, v - 2^{l+1}\}$ of $E(\mathcal{H}_v^l)$ and the edge $\{v - 2^l, v + 2^l\}$ of $E(\mathcal{B}_v^l)$ from

the edges of $\text{link}_{\Delta_{v-2}}(v-1)$, as in the cases $v = 2, 3$; otherwise $\text{link}_{\Delta_{v-1}}(v)$ is isomorphic to $\text{link}_{\Delta_{v-2}}(v-1)$, as in the case $v = 4$. To get the set (3.2), we pose

$$j(v) = \min\{l \in \mathbb{N} : 2^{l-1} \leq v-1 < 2^l\}$$

for $2 \leq v \leq 2^m - 1$. Moreover we have $1 \leq v-1 < 2^m$, so that $j(v) \leq m$, and $\mathcal{F}(\text{link}_{\Delta_{v-1}}(v))$ is equal to

$$E(\mathcal{H}_v^{j(v)}) \cup E(\mathcal{B}_v^{j(v)}) \cup \{v - 2^m + 1, v + 1\} \cup E(\mathcal{G}_v) \quad (3.2)$$

that is connected because \mathcal{G} and \mathcal{H} are, and they are joined by $\{v - 2^m + 1, v + 1\}$. Now, we consider the vertices v in $2^m + 1, 2^m + 2, \dots, 2^{m+1} - 1$.

For $v = 2^m + 1$, since 2^m has not been removed, $2^m = v - 1 \in V(\text{link}_{\Delta_{v-2}}(v))$ (see Figure 6), and we have

$$\mathcal{F}(\text{link}_{\Delta_{v-2}}(v)) = \{v - 1, v + 1\} \cup \{v - 1, v + 2^m - 1\} \cup E(\mathcal{G}_v).$$

We exploit the steps $v = 2^m + 2, 2^m + 3$ before giving the general set (3.3) for $2^m + 2 \leq v \leq 2^{m+1} - 2$.

For $v = 2^m + 2$, since 2^m has not been removed, $2^m = v - 2 \in V(\text{link}_{\Delta_{v-2}}(v))$ (see Figure 7) and

$$\mathcal{F}(\text{link}_{\Delta_{v-2}}(v)) = \{v - 2, v + 2\} \cup E(\mathcal{G}_v).$$

For $v = 2^m + 3$, v is not adjacent to 2^m (since $3 \notin \bar{S}$) and since we removed the vertex w for $1 \leq w \leq 2^m - 1$,

$$\mathcal{F}(\text{link}_{\Delta_{v-2}}(v)) = E(\mathcal{G}_v)$$

(see Figure 8). In general, for $2^m + 2 \leq v \leq 2^{m+1} - 2$, since the only vertex in $\{1, \dots, 2^m\}$ that we have not removed is 2^m , when $v = 2^m + 2^j$, then $v - 2^j \in V(\text{link}_{\Delta_{v-2}}(v))$, that is

$$\mathcal{F}(\text{link}_{\Delta_{v-2}}(v)) = \begin{cases} \{v - 2^j, v + 2^j\} \cup E(\mathcal{G}_v) & \text{if } v = 2^m + 2^j \\ E(\mathcal{G}_v) & \text{otherwise.} \end{cases} \quad (3.3)$$

For $v = 2^{m+1} - 1$, since v is adjacent to 2^m , we have

$$\mathcal{F}(\text{link}_{\Delta_{v-2}}(v)) = \{v - 2^m + 1, v + 1\} \cup E(\mathcal{G}_v)$$

(see Figure 10). To complete the decomposition, we need to remove the vertices v in $2^{m+1} + 1, \dots, n - 1$.

For $v = 2^{m+1} + 1$, since 2^{m+1} has not been removed, $2^{m+1} = v - 1 \in V(\text{link}_{\Delta_{v-3}}(v))$. On the other hand, $1 = v + 2^m$ has been removed (Figure 11), hence

$$\mathcal{F}(\text{link}_{\Delta_{v-3}}(v)) = \{v + 2^m - 1, v - 1\} \cup \{v - 1, v + 1\} \cup E(\mathcal{P}_v^{m-1}).$$

For $2^{m+1} + 2 \leq v \leq n - 2 = 2^{m+1} + 2^m - 2$, we set

$$k(v) = \min\{l \in \mathbb{N} : 2^l \leq n - v < 2^{l+1}\}.$$

We observe that the path $\mathcal{P}_v^{k(v)}$ is contained in $\text{link}_{\Delta_{v-3}}(v)$ (see Figure 12 and Figure 13). Moreover, if $v = 2^{m+1} + 2^j$, then v is adjacent to 2^{m+1} , and $\text{link}_{\Delta_{v-3}}(v)$ contains the edge $\{v - 2^j, v + 2^j\}$ (see Figure 12). Hence, for $2^{m+1} + 2 \leq v \leq n - 2$ we have

$$\mathcal{F}(\text{link}_{\Delta_{v-3}}(v)) = \begin{cases} \left\{ \{v - 2^j, v + 2^j\} \cup E(\mathcal{P}_v^{k(v)}) \right\} & \text{if } v = 2^{m+1} + 2^j \\ E(\mathcal{P}_v^{k(v)}) & \text{otherwise.} \end{cases}$$

The existence of $\{v - 2^j, v + 2^j\}$ is guaranteed by the inequality $2^j \leq 2^{k(v)}$. In fact, $n - v = 2^{m+1} + 2^m - 2^{m+1} - 2^j = 2^m - 2^j$. Since $j \leq m - 1$, then

$$2^{m-1} \leq 2^m - 2^j \leq 2^m.$$

So $k(v) = m - 1$ for $v = 2^{m+1} + 2^j$.

The last vertex that we remove is $v = n - 1$. We have not removed yet $2^{m+1} = v - 2^m + 1$ and $0 = v + 1$. Therefore its link is formed only by the edge $\{v - 2^m + 1, v + 1\}$ (see Figure 14). The only vertices that we have not removed are $\{0, 2^m, 2^{m+1}\}$, that is a simplex on 3 vertices. The assertion follows. \square

Now we give an example of circulant belonging to the class above.

Example 2. *The first circulant of the class is $G = C_{24}(\overline{1, 2, 4, 7, 8})$ and let $\Delta = \Delta(G)$. We want to prove that the sequence*

$$\mathcal{M} = [1, 2, \dots, \widehat{8}, 9, \dots, \widehat{16}, 17, \dots, 23]$$

satisfies the conditions (1) and (2) of Lemma 3.

By using the notation of the proof of Theorem 1, we have

$$\begin{aligned} E(\mathcal{H}_v^1) &= \left\{ \{v - 2, v - 4\}, \{v - 4, v - 8\}, \{v - 8, v - 7\} \right\}, \\ E(\mathcal{G}_v) &= \left\{ \{v + 1, v + 2\}, \{v + 2, v + 4\}, \{v + 4, v + 8\}, \{v + 8, v + 1\}, \{v + 8, v + 7\} \right\}, \\ E(\mathcal{P}_v^2) &= \left\{ \{v + 1, v + 2\}, \{v + 2, v + 4\} \right\}, \\ E(\mathcal{P}_v^1) &= \left\{ \{v + 1, v + 2\} \right\}, \\ E(\mathcal{B}_v^1) &= \left\{ \{v - 2, v + 2\}, \{v - 4, v + 4\}, \{v - 8, v + 8\} \right\}. \end{aligned}$$

For $v = 1$, $\text{link}_{\Delta}(v)$ is represented in Figure 2 and it is 1-dimensional and connected. From now on, we omit last observation that will be clear by the figures.

Then we define $\Delta_1 = \text{del}_{\Delta}(1)$ and we study $\text{link}_{\Delta_{v-1}}(v)$ when $v = 2$. Since $v - 1 = 1$ and we have removed the vertex 1, $v - 1$ does not appear in $\text{link}_{\Delta_{v-1}}(v)$ (Figure 3). We point out that the latter is formed by \mathcal{H}_v^1 , \mathcal{B}_v^1 , \mathcal{G}_v and the edge $\{v - 7, v + 1\}$.

Then we set $\Delta_2 = \text{del}_{\Delta}(2)$ and we look at $\text{link}_{\Delta_{v-1}}(v)$ when $v = 3$. Since we removed $1 = v - 2$ and $2 = v - 1$, then $v - 1, v - 2 \notin \text{link}_{\Delta_{v-1}}(v)$ (Figure 4).

For $v = 4$, the same facts of the case $v = 3$ hold. Hence $\text{link}_{\Delta_{v-1}}(v)$ is isomorphic to

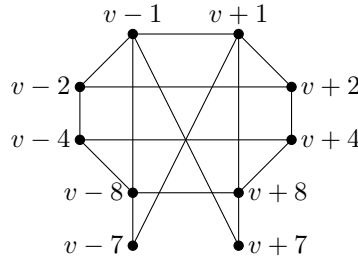


Figure 2: $\text{link}_{\Delta}(v)$ for any $v \in V(G)$.

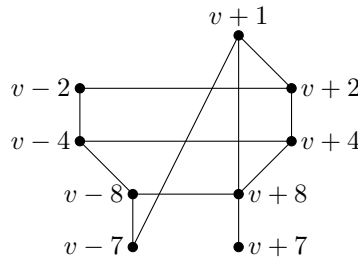


Figure 3: $\text{link}_{\Delta_{v-1}}(v)$ for $v = 2$.

$\text{link}_{\Delta_{v-2}}(v-1)$ (Figure 4).

For $v = 5$, since we have removed $1 = v - 4$, $3 = v - 2$ and $4 = v - 1$, then they do not appear in $\text{link}_{\Delta_{v-1}}(v)$ (Figure 5).

For $v = 6$ and 7 , the same facts of the case $v = 5$ hold and their links are isomorphic to the one in Figure 5.

Now we jump from $v = 7$ to $v = 9$, without removing the vertex 8 . It implies that we have $v - 1 = 8$ is in $\text{link}_{\Delta_{v-2}}(v)$ (Figure 6).

From $v = 10$ to $v = 14$, $\text{link}_{\Delta_{v-2}}(v)$ is formed by \mathcal{G}_v (Figure 8), and the edge connecting 8 to \mathcal{G}_v , when $v = 8 + 2^j$, namely $\{8 = v - 2^j, v + 2^j\}$ (Figure 7 and Figure 9).

For $v = 15$, $8 = v - 7$ appears in $\text{link}_{\Delta_{v-2}}(v)$ and once again it is connected to \mathcal{G}_v (Figure 10).

We jump from $v = 15$ to $v = 17$. Since we have not removed $16 = v - 1$, then $v - 1$ is contained in $\text{link}_{\Delta_{v-3}}(v)$ (analogously to the case $v = 9$). On the other hand, since we have removed $1 = v + 8$, it does not appear in $\text{link}_{\Delta_{v-3}}(v)$ (Figure 11), that is the path \mathcal{P}_v^2 plus the edges $\{v + 1, v - 1\}$ and $\{v - 1, v + 7\}$.

From $v = 18$ to $v = 20$, $\text{link}_{\Delta_{v-3}}(v)$ is formed by \mathcal{P}_v^2 (Figure 13), and the edge connecting 16 to \mathcal{P}_v^2 , when $v = 16 + 2^j$, namely $\{16 = v - 2^j, v + 2^j\}$ (e.g. Figure 12).

For $v = 21, 22$, they are not adjacent to 16 and we have removed $1, 2 = v + 4$. That is $\text{link}_{\Delta_{v-3}}(v)$ is only \mathcal{P}_v^1 .

For $v = 23$, the only vertices not yet removed are $0 = v + 1$ and $16 = v - 7$ (Figure 14).

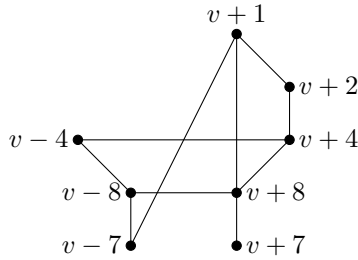


Figure 4: $\text{link}_{\Delta_{v-1}}(v)$ for $v = 3, 4$.

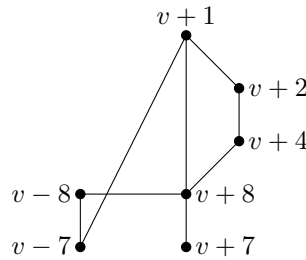


Figure 5: $\text{link}_{\Delta_{v-1}}(v)$ for $v = 5, 6, 7$.

Hence we have proved (ii).(1) of Lemma 3. Then we are left with the only triangle $\{0, 8, 16\}$, that is a simplex on 3 vertices, so that (ii).(2) of Lemma 3 is satisfied. Hence Δ is vertex decomposable.

4 Level and Gorenstein algebras

In this section we prove that any 2-dimensional vertex decomposable independence complex of circulants has a level Stanley-Reisner ring.

Definition 1. We say that a graph G is l -connected if for every subset $S \subseteq V(G)$ of cardinality $|S| < l$, then G restricted to the set of vertices $V \setminus S$ is connected. We simply call connected graph a 1-connected graph, biconnected graph a 2-connected graph, triconnected graph a 3-connected graph.

Lemma 4. Let $G = C_n(a, b)$ be a connected circulant graph. Then G is triconnected.

Proof. The graph G is connected if and only if $\text{gcd}(n, a, b) = 1$. We have to prove that after we remove any two vertices the graph remains connected. We take out the vertex 0. Since any connected circulant is biconnected, the remaining graph is connected. Let $G^* = G \setminus \{0\}$. We have two cases:

(T1) One of the elements a, b is coprime with n .

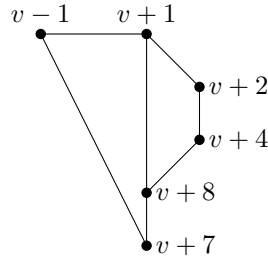


Figure 6: $\text{link}_{\Delta_{v-2}}(v)$ for $v = 9$.

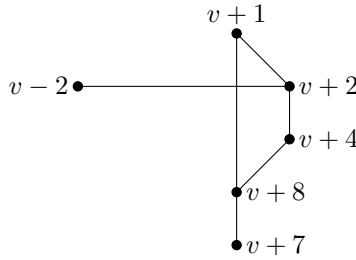


Figure 7: $\text{link}_{\Delta_{v-2}}(v)$ for $v = 10$.

(T2) Neither a nor b is coprime with n .

(T1) We assume a coprime with n , then

$$V(G) = \{a, 2a, \dots, (n-1)a\}.$$

Let $b = sa$. By removing the vertex a (respectively the vertex $(n-1)a$), the remaining graph is connected through the path $\{2a, 3a, \dots, (n-1)a\}$ (respectively $\{a, 2a, \dots, (n-2)a\}$). So we need to consider the removal of ia with $2 \leq i \leq n-2$. We end up with the two paths on vertices

$$A = \{a, \dots, (i-1)a\}, \quad B = \{(i+1)a, \dots, (n-1)a\}.$$

We prove that the two paths above are connected each other by some edges. We have two cases: $i \leq s, i > s$. If $i \leq s$, then $i+1 \leq s+1 \in B$ and $\{a, a+b\} = \{a, (s+1)a\} \in E(G)$. If $i > s$, since $1 < s \leq i-1$, then $(i+1-s)a \in A$. Hence $\{(i+1-s)a, (i+1)a\} \in E(G)$. The assertion follows

(T2) We assume $d = \gcd(b, n)$ and since $\gcd(a, b, n) = 1$, then a is coprime with d . Let $n = ld$. It follows that the vertex set $V(G^*)$ can be partitioned in

$$\begin{aligned} V_0 &= \{d, \dots, (l-1)d\}, \\ V_1 &= \{a, a+d, \dots, a+(l-1)d\}, \\ &\vdots \\ V_{d-1} &= \{(d-1)a, (d-1)a+d, \dots, (d-1)a+(l-1)d\}. \end{aligned}$$

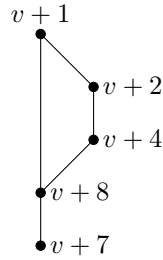


Figure 8: $\text{link}_{\Delta_{v-2}}(v)$ for $v = 11, 13, 14$.

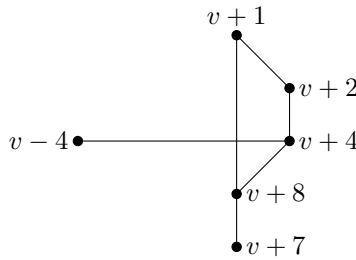


Figure 9: $\text{link}_{\Delta_{v-2}}(v)$ for $v = 12$.

We observe that the sets V_i and V_{i+1} are connected each other since $a \in S$. Moreover each V_i is connected, since $d \mid b \in S$, and if $r \neq 0$, then V_r is a cycle and it is biconnected. It implies that after removing a vertex from V_r with $r \neq 0$, the graph remains connected. So we assume $r = 0$, and we remove the vertex kd for some k . Hence we have to prove that the two sets

$$V'_0 = \{d, \dots, (k-1)d\}, \quad V''_0 = \{(k+1)d, \dots, (l-1)d\}$$

are connected each other. Take $x \in V'_0$ and $y \in V''_0$. Then $a+x, a+y \in V_1$. Since V_1 is connected, the assertion follows. \square

We present an interesting observation on the reduced Euler characteristic of 2-dimensional complexes for circulants.

Lemma 5. *Let $n \geq 6$ and $G = C_n(S)$ be circulant graph $\dim \Delta = 2$. Then*

$$\tilde{\chi}(\Delta) \neq 0.$$

Proof. If $n = 6$ it is an easy task. We consider the case $n > 6$. The \mathbf{f} -vector of Δ is given by (f_0, f_1, f_2) and

$$\tilde{\chi}(\Delta) = -1 + f_0 - f_1 + f_2 = -1 + n - f_1 + f_2.$$

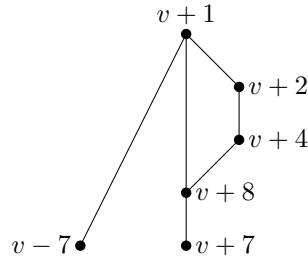


Figure 10: $\text{link}_{\Delta_{v-2}}(v)$ for $v = 15$.

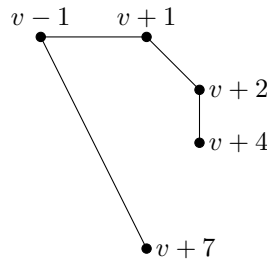


Figure 11: $\text{link}_{\Delta_{v-3}}(v)$ for $v = 17$.

Let $d = \gcd(f_2, f_1, n)$. By Lemma 2.2 of [15] it follows that

$$f_1 = \frac{nf_{1,0}}{2}, \quad f_2 = \frac{nf_{2,0}}{3}$$

Hence in any case $d \in \{\frac{n}{6}, \frac{n}{3}, \frac{n}{2}, n\}$. Since $n > 6$, then $d > 1$. Therefore it follows that

$$\tilde{\chi}(\Delta) \equiv -1 \pmod{d}$$

and $\tilde{\chi}(\Delta) \neq 0$. □

Lemma 6. *Let $G = C_n(\bar{S})$ be a circulant graph such that Δ is 2-dimensional and vertex decomposable. Then $|S| \geq 2$.*

Proof. By contraposition, let us assume that $G = C_n(a)$, for $a \in \mathbb{Z}_n$. Since $\dim \Delta = 2$, then $n = 3a$. It implies that Δ is disconnected, that implies Δ is not vertex decomposable. □

Proposition 5. *Let $G = C_n(\bar{S})$ be a circulant graph such that Δ is 2-dimensional and Cohen-Macaulay. Then*

$$\text{reg } R/I(G) = 3.$$

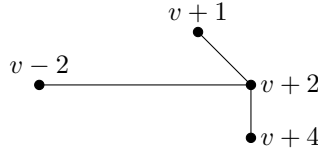


Figure 12: $\text{link}_{\Delta_{v-3}}(v)$ for $v = 18$.

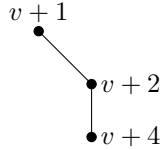


Figure 13: $\text{link}_{\Delta_{v-3}}(v)$ for $v = 19$.

Proof. Since $R/I(G)$ is Cohen-Macaulay, then by Lemma 5, [16, Remark 1.2] and [7, Corollary 4.8] we get

$$\text{reg } R/I(G) = \text{depth } R/I(G) = \dim R/I(G) = 3.$$

□

Theorem 3. *Let $G = C_n(\bar{S})$ be a circulant graph such that Δ is 2-dimensional and vertex decomposable. Then $R/I(G)$ is a level algebra.*

Proof. From Remark 5, $\tilde{\chi}(\Delta)$ is always non-zero. Hence according to Lemma 2, we have to prove that the Cohen-Macaulay type of $R/I(G)$ coincides with $\tilde{\chi}(\Delta)$. Namely, we have to compute the last total Betti number of the minimal free resolution of $R/I(G)$. Since Δ is also Cohen-Macaulay, then

$$\text{depth } R/I(G) = \dim R/I(G) = 3.$$

From Auslander-Buchsbaum formula, we have that

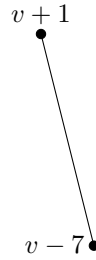
$$\text{pd } R/I(G) = \dim R - \text{depth } R/I(G) = n - 3.$$

From Proposition 5 we have $\text{reg } R/I(G) = 3$. So we have to look at the Betti numbers $\beta_{n-3,j}$ for $j \in \{1, 2, 3\}$. According to Hochster's Formula (Theorem 2),

$$\beta_{i,\sigma} = \dim_K \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma|}; K).$$

First of all, we assume $|\sigma| = n - 2$ and so

$$\beta_{n-3,n-2} = \dim_K \tilde{H}_0(\Delta_{|\sigma|}; K) = 0$$

Figure 14: $\text{link}_{\Delta_{v-3}}(v)$ for $v = 23$.

from Lemma 6 and Lemma 4, and from Equation (1.2). Now we assume $|\sigma| = n - 1$, and so

$$\beta_{n-3, n-1} = \dim_K \tilde{H}_1(\Delta_{|\sigma}; K).$$

Since $\Delta_{|\sigma}$ is the simplicial complex defined on $V(G) = \{0, 1, 2, \dots, \hat{x}, \dots, n - 1\}$, it holds

$$\Delta_{|\sigma} \simeq \text{del}_{\Delta}(x)$$

that is vertex decomposable because Δ is, and hence Cohen-Macaulay. Therefore $\beta_{n-3, n-1} = 0$. Finally, we assume $|\sigma| = n$, and hence

$$\beta_{n-3, n} = \dim_K \tilde{H}_2(\Delta; K) = \tilde{\chi}(\Delta),$$

and the assertion follows. \square

Corollary 1. *Let $G = C_n(\overline{1, 2, 4, \dots, 2^m, 2^m - 1})$ with $m \geq 3$ and $n = 3 \cdot 2^m$. Then $R/I(G)$ is a level algebra.*

It is of interest to know whether a level algebra is also a Gorenstein algebra. In general, we have the following

Theorem 4. *Let G be a non-empty circulant graph with $\dim R/I(G) = 3$. The following are equivalent:*

1. $R/I(G)$ is Gorenstein;
2. $G = C_6(3)$.

Proof. (2) \Rightarrow (1). It is easy to verify that $\Delta(C_6(3))$ is vertex decomposable. Then, according to Theorem 3 and Lemma 2, we have to compute $\tilde{\chi}(\Delta)$. The f -vector of Δ is $(1, 6, 12, 8)$, so $\tilde{\chi}(\Delta) = -1 + 6 - 12 + 8 = 1$. Therefore, the assertion follows.

(1) \Rightarrow (2). A necessary condition for $R/I(G)$ to be Gorenstein is that its h -vector has to be symmetric (see [21, Corollary 5.3.10]). From Proposition 3, by requiring that

$$h_1 = h_2,$$

we obtain

$$n(s - 3) = -6.$$

The pairs of integers with $n > 2$ that satisfy the equation above are

$$(3, 1), (6, 2).$$

The first pair is not admissible because $\Delta(G)$ will be the simplex on 3 vertices and G the empty graph. So the h -vector has $h_1 = h_2$ if $n = 6$ and $s = 2$. So the only candidates for G are

$$C_6(1), C_6(2), C_6(3).$$

In the first case $\Delta(G)$ is not pure, hence $R/I(G)$ cannot be Cohen-Macaulay. In the second case $\dim R/I(G) = 2$. The assertion follows. \square

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