THE STANLEY–REISNER IDEAL OF THE ROOK COMPLEX OF POLYOMINOES

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Abstract. We study the properties of the rook complex $\mathcal{R}$ of a polyomino $\mathcal{P}$ seen as independence complex of a graph G, and the associated Stanley-Reisner ideal $I_{\mathcal{R}}$. In particular, we characterise the polyominoes $\mathcal{P}$ having a pure rook complex, and the ones whose Stanley-Reisner ideal has linear resolution. Furthermore, we prove that for a class of polyominoes the Castelnuovo-Mumford regularity of $I_{\mathcal{R}}$ coincides with the induced matching number of G.

1. Introduction

Polyominoes are two-dimensional objects obtained by joining edge by edge squares of same size. Originally, polyominoes appeared in mathematical recreations [6], but it turned out that they have applications in various fields, for example, theoretical physics and bio-informatics. Among the most popular topics in combinatorics related to polyominoes one finds enumerating polyominoes of given size, including the asymptotic growth of the numbers of polyominoes, tiling problems, and reconstruction of polyominoes. The actual research on polyominoes under an algebraic point of view focuses on the study of the polyomino ideal, a quadratic binomial ideal associated to the geometry of polyominoes (see [12, 14, 10, 11, 2, 15, 13, 3]). In the last three papers, the authors compute some algebraic invariants of the polyomino ideal by studying the rook polynomial $\sum_{i=1}^{n} r_i t^i$, i.e. the polynomial whose coefficient r_i represents the number of distinct ways of arranging i rooks on squares of a polyomino $\mathcal{P}$ in non-attacking positions. The degree of such polynomial is called *rook number* and it is denoted by $r(\mathcal{P})$. The rook arrangements described above give rise to a simplicial complex, called *rook complex.* In this paper, by focusing on the rook complex, we study polyominoes under a monomial point of view, as described below.

Let Δ be a simplicial complex on vertices $\{1, \dots, n\}$ and let $R = K[x_1, \dots, x_n]$ be the polynomial ring on n variables over a field K. The *Stanley-Reisner ideal* or face ideal, denoted by I_Δ, is known to be the ideal generated by the square-free monomials $\{x_{i_1}, \dots, x_{i_r}\}$ such that $\{i_1, \dots, i_r\} \notin \Delta$. Let G be a graph on vertices $\{1, \dots, n\}$ and let $R = K[x_1, \dots, x_n]$ be the polynomial ring on n variables over a field K. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Edge ideals of graphs have been introduced by Villarreal [17] in 1990, where he studied the Cohen–Macaulay property of such ideals. Many authors have focused their attention on such ideals (e.g.[7], [4]).

The two above concepts have a nice relationship. If Δ is the independence complex of G, i.e. the simplicial complex of the independent sets of G, then it holds $I(G) = I_\Delta$. For such a reason, it is reasonable to study the Stanley-Reisner ideal of the rook complex of polyominoes. Let $\mathcal{P}$ be a polyomino, let $\mathcal{R}$ be its rook complex and let $I_{\mathcal{R}}$ be the Stanley-Reisner ideal of $\mathcal{R}$. Let $G_{\mathcal{P}}$ be the graph on the cells of $\mathcal{P}$ having $\mathcal{R}$ as independence complex. It follows that $V(G_{\mathcal{P}}) = \{C\}_{C \in \mathcal{P}}$ and

$$E(G_{\mathcal{P}}) = \{\{C, D\} : C \text{ and } D \text{ lie on the same row or column}\}.$$

Some challenging problems in the modern research are the classification of Cohen-Macaulay rings and the study of the minimal free resolution and Castelnuovo-Mumford regularity. In Section 3, we characterise the polyominoes having a pure rook complex, i.e. all the maximal faces have the same cardinality, because the pureness is a necessary condition for the Cohen-Macaulayness. For the aim of studying minimal free resolution, in Section 4 we characterise the polyominoes for which

$I(G_{\mathcal{P}})$ has linear resolution in terms of the chordality of the complement graph $\bar{G}_{\mathcal{P}}$ (see Theorem 2.1). We call such polyominoes *brush* polyominoes because of their nice structure: a long interval (the handle) with dominoes as bristles. In Section 5 we consider brush polyominoes with longer bristles and for this class we prove that the Castelnuovo-Mumford regularity of $I(G_{\mathcal{P}})$ coincides with the induced matching number of $G_{\mathcal{P}}$.

2. Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article.

2.1. **Polyominoes.**

In this subsection, we recall general definitions and notation on polyominoes.

Let $a = (i, j), b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$. The set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an *interval* of $\mathbb{N}^2$. Moreover, if $i < k$ and $j < \ell$, then $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$. In particular, a, b are the *diagonal corners* and c, d are the *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the left lower (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. The corners of C are called the vertices of C. The set of vertices of C is denoted by $V(C)$. The edge set of C, denoted by $E(C)$, is

$$\{\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}\}.$$

We denote by $\ell(C)$, the left lower corner of a cell C.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected*, if there is a sequence of cells $C = C_1, \dots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \dots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \dots, C_m$ is called a *path* (connecting C and D). A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P}) = \cup_{C\in\mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$ and by $E(\mathcal{P}) = \cup_{C\in\mathcal{P}} E(C)$ the edge set of $\mathcal{P}$. In particular, a polyomino could be also seen as a connected bipartite graph. Note that, if $a, b \in V(\mathcal{P})$, then a and b are connected in $V(\mathcal{P})$ by a path of edges. More precisely, one can find a sequence of vertices $a = a_1, \dots, a_n = b$ such that $\{a_i, a_{i+1}\} \in E(\mathcal{P})$, for all $i = 1, \dots, n - 1$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and we denote it by $\operatorname{rk}\mathcal{P}$. We also define the *lower left corner* of $\mathcal{P}$ as $\ell(\mathcal{P}) = \min\{\ell(C) : C \in \mathcal{P}\}$. Each proper interval $[(i, j), (k, l)]$ in $\mathbb{N}^2$ can be identified as a polyomino and it is referred to as *rectangular* polyomino, or simply as rectangle. If $s = k - i$ and $t = l - j$ we say that the rectangle has *size* $s \times t$. In particular, given a rectangle of $\mathcal{P}$ we call *diagonal cells* the cells A, B such that $\ell(A) = (i, j)$ and $\ell(B) = (k - 1, l - 1)$ and *antidiagonal cells* the cells C, D such that $\ell(C) = (i, l - 1)$ and $\ell(D) = (k - 1, j)$.

A polyomino $\mathcal{P}$ is called a *subpolyomino* of $\mathcal{P}'$, if all cells of $\mathcal{P}$ are contained in $\mathcal{P}'$. Given a polyomino $\mathcal{P}$, the smallest rectangle (with respect to its size) containing $\mathcal{P}$ as a subpolyomino, is called the *bounding box* of $\mathcal{P}$.

We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \dots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \dots, m$. Roughly speaking, a polyomino without a "hole" is called a simple polyomino. We say that a polyomino $\mathcal{P}$ is *thin* if $\mathcal{P}$ does not contain the square tetromino (see Figure 1) as a subpolyomino.

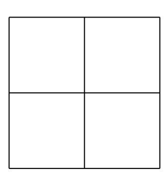


Figure 1. The square tetromino

An interval $[a,b]$ with $a=(i,j)$ and $b=(k,\ell)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j=\ell$ and the sets $\{(r,j),(r+1,j)\}$ for $r=i,\dots,k-1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it *maximal horizontal edge interval*. Similarly, one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$. A polyomino $\mathcal{P}$ is called *row convex* if for any two of its cells with lower left corners $a=(i,j)$ and $b=(k,j)$, with $k>i$, all cells with lower left corners (l,j) with $i\leq l\leq k$ are cells of $\mathcal{P}$. Similarly, $\mathcal{P}$ is called *column convex* if for any two of its cells with lower left corners $a=(i,j)$ and $b=(i,k)$, with $k>j$, all cells with lower left corners (i,l) with $j\leq l\leq k$ are cells of $\mathcal{P}$. If a polyomino $\mathcal{P}$ is simultaneously row and column convex then $\mathcal{P}$ is called *convex*. Let $\mathcal{C}: C_1,C_2,\dots,C_m$ be a path of cells and (i_k,j_k) be the lower left corner of C_k for $1\leq k\leq m$. Then $\mathcal{C}$ has a change of direction at C_k for some $2\leq k\leq m-1$ if $i_{k-1}\neq i_{k+1}$ and $j_{k-1}\neq j_{k+1}$. A convex polyomino $\mathcal{P}$ is called k-convex if any two cells in $\mathcal{P}$ can be connected by a path of cells in $\mathcal{P}$ with at most k change of directions. A *cell interval* is a path of cells with no change of direction. We recall the construction of the rook complex $\mathcal{R}_{\mathcal{P}}$. We identify rooks on the cells of a polyomino with the cells themselves. Two cells C and D of a polyomino $\mathcal{P}$ are called *attacking* if they belong to the same cell interval. Otherwise, we say that they are *non-attacking*. The elements of $\mathcal{R}_{\mathcal{P}}$ are subsets of $\mathcal{P}$ containing pairwise non-attacking rooks (including $\varnothing$ and the singletons $\{C\}$ for any $C\in\mathcal{P}$).

2.2. **Graphs and simplicial complexes.** Set $V=\{x_1,\dots,x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that: 1) $\{x_i\}\in\Delta$ for all $x_i\in V$; 2) $F\in\Delta$ and $G\subseteq F$ imply $G\in\Delta$. An element $F\in\Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.
The dimension of a face $F\in\Delta$ is $\dim F=|F|-1$, and the dimension of Δ is the maximum of the dimensions of all facets. Moreover, if all the facets of Δ have the same dimension, then we say that Δ is *pure*. Let $d-1$ be the dimension of Δ and let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1}=1$. Then the f-vector of Δ is the $(d+1)$-tuple $f(\Delta)=(f_{-1},f_0,\dots,f_{d-1})$. The h-vector of Δ is $h(\Delta)=(h_0,h_1,\dots,h_d)$ with

$$h_k=\sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{k-i}f_{i-1}. \tag{1}$$

Similarly, one can express the entries of f-vector by the entries of the h-vector, in fact for $i=0,\dots,d$

$$f_{i-1}=\sum_{k=0}^{i}\binom{d-k}{i-k}h_k \tag{2}$$

It follows that $f(\mathcal{R})=(f_{-1},\dots,f_{d-1})$ where $f_{i-1}=r_i$ and $d=r(\mathcal{P})$. We recall the following definitions:

$$\operatorname{link}_\Delta(F)=\{G\in\Delta:\ F\cap G=\varnothing\ \text{ and } F\cup G\in\Delta\},\ \operatorname{del}_\Delta(F)=\{G\in\Delta: F\cap G=\varnothing\}.$$

We define *chain complex* as follows:

$$\mathcal{C}\ :\ 0\to K^{f_{d-1}}\xrightarrow{\partial_{d-1}}K^{f_{d-2}}\xrightarrow{\partial_{d-2}}\dots\xrightarrow{\partial_0}K\to 0$$

and by definition the $i-$th *reduced homology group* $\widetilde{H}_i(\Delta;K)$ is

$$\widetilde{H}_i(\Delta;K)=\ker(\partial_i)/\operatorname{im}(\partial_{i+1}).$$

Let Δ be a pure independence complex of a graph G. We say that Δ is *vertex decomposable* if one of the following conditions hold: (1) $n=0$ and $\Delta=\{\varnothing\}$; (2) Δ has a unique maximal facet $\{x_0,\dots,x_{n-1}\}$; (3) There exists $x\in V(G)$ such that both $\operatorname{link}_\Delta(x)$ and $\operatorname{del}_\Delta(x)$ are vertex decomposable and the facets of $\operatorname{del}_\Delta(x)$ are also facets in Δ.
We say that Δ is *Cohen-Macaulay* if for any $F\in\Delta$ we have that $\dim_K\widetilde{H}_i(\operatorname{link}_\Delta(F),K)=0$ for any

$i < \dim \operatorname{link}_\Delta(F)$. In particular, Δ is Cohen-Macaulay if and only if R/I_Δ is a Cohen-Macaulay ring (see [1]). It is well known that

$$\Delta \text{ Vertex Decomposable} \Rightarrow \Delta \text{ Cohen-Macaulay } \Rightarrow \Delta \text{ Pure}.$$

Let $\mathbb{F}$ be the minimal free resolution of $R/I(G)$. Then

$$\mathbb{F} \;:\; 0 \to F_p \to F_{p-1} \to \ldots \to F_0 \to R/I(G) \to 0$$

where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$. The $\beta_{i,j}$ are called the *Betti numbers* of $\mathbb{F}$. For any i, $\beta_i = \sum_j \beta_{i,j}$ is called the i-th *total Betti number*. The *Castelnuovo-Mumford regularity* of $R/I(G)$, denoted by reg $R/I(G)$ is defined as

$$\operatorname{reg} R/I(G) = \max\{j - i : \beta_{i,j} \neq 0\}.$$

Let G be a graph. A collection C of edges in G is called an *induced matching* of G if the edges of C are pairwise disjoint and the graph having C as edge set is an induced subgraph of G. The maximum size of an induced matching of G is called the *induced matching number* of G and we denote it by $\nu(G)$. The *complement graph* $\bar{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the non-edges of G. We conclude the section by stating some known results relating chordality and induced matching number to the Castelnuovo-Mumford regularity. The first one is due to Fröberg ([5, Theorem 1])

Theorem 2.1. *Let G be a graph. Then* $\operatorname{reg} R/I(G) \leq 1$ *if and only if $\bar{G}$ is chordal.*

The second one is due to Katzman ([9, Lemma 2.2]).

Theorem 2.2. *For any graph G, we have* $\operatorname{reg} R/I(G) \geq \nu(G)$.

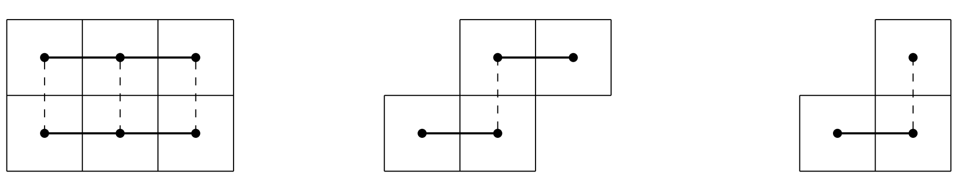
3. Pureness of $\mathcal{R}$

In this section, we characterize the polyominoes having a pure rook complex. For this aim, in the following definition, we introduce partitions on polyominoes. From now on, given two cell intervals I and J, we write $I \cap J$ to denote the common cells of I and J. We denote by $\mathcal{C}$ the set of all maximal cell intervals of $\mathcal{P}$.

Definition 3.1. Let $\mathcal{P}$ be a polyomino. A subset $\varnothing \neq \mathcal{A} \subset \mathcal{C}$ is called a *partition* of $\mathcal{P}$ if

(1) $\forall I, J \in \mathcal{A}$ we have $I \cap J = \varnothing$;
(2) $\bigcup_{I \in A} I = \mathcal{P}$.

Example 3.2. A polyomino admits at most two partitions, one horizontal and one vertical. In Figure 2, the polyomino $\mathcal{P}_1$ admits two partitions, the polyomino $\mathcal{P}_2$ admits one partition and the polyomino $\mathcal{P}_3$ admits no partition.



$\mathcal{P}_1$ $\mathcal{P}_2$ $\mathcal{P}_3$

Figure 2. We highlight with a thick line the horizontal cell intervals and with a dashed line the vertical cell intervals

Definition 3.3. A cell interval $I = C_1 C_2 \cdots C_m \in \mathcal{C}$ is called *embedded* if there exists $F = \{D_1, \ldots, D_m\} \in \mathcal{R}$ such that for any $i \in \{1, \ldots, m\}$ the set $\{C_i, D_i\}$ is attacking.

Remark 3.4. Let I be a non-embedded interval. Then any facet $F \in \mathcal{R}_\mathcal{P}$ is such that $F \cap I \neq \varnothing$.

Definition 3.5. Let $\mathcal{A}$ be a partition of $\mathcal{P}$. If no interval of $\mathcal{A}$ is embedded then $\mathcal{A}$ is called *super partition.*

Example 3.6. We consider the polyominoes in Figure 2. The partition of the polyomino $\mathcal{P}_2$ and the horizontal partition of $\mathcal{P}_1$ are super partitions, while the vertical one is not a super partition because any vertical cell interval is embedded. In particular referring to Figure 3, $\{E, C\}$ embeds the interval AD, $\{D, C\}$ embeds the interval BE and $\{D, B\}$ embeds the interval CF.

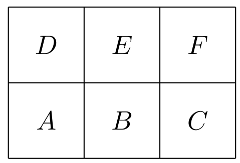


FIGURE 3. All of the vertical intervals of $\mathcal{P}_1$ are embedded

We have seen that polyominoes could have at most two partitions, moreover we have the following.

Proposition 3.7. *A polyomino $\mathcal{P}$ has two super partitions if and only if $\mathcal{P}$ is a square.*

Proof. If $\mathcal{P}$ is the $n \times n$ square, then it has two partitions given by the n rows and the n columns. Moreover, any interval can be attacked at most on $n-1$ cells, that is, the two partitions have no embedded intervals.

Conversely, let $\mathcal{A}$ and $\mathcal{B}$ be two super partitions of $\mathcal{P}$. Let $I_1 = C_1C_2 \cdots C_n \in \mathcal{A}$ with a free edge interval, without loss of generality the uppermost one. Let $J_1, \ldots, J_n \in \mathcal{B}$ be such that $I_1 \cap J_i = \{C_i\}$. Let $m = \min_i\{|J_i|\}$, $t \in \{1, \ldots, n\}$ be such that $|J_t| = m$ and let $I_1, \ldots, I_m \in \mathcal{A}$ be the rows covering the intervals $J_1, \ldots, J_n$. We claim $m = n$.
If $m < n$, for any subset $\{i_1, \ldots, i_m\} \subset \{1, \ldots, n\} \setminus \{t\}$ it holds that $F = \{D_1, \ldots, D_m\} \in \mathcal{R}$, where $D_k = I_k \cap J_{i_k}$. Namely, J_t is embedded and $\mathcal{B}$ is not a super partition of $\mathcal{P}$.
If $m > n$, for any $k \in \{1, \ldots, n\}$ let $D_k = I_{k+1} \cap J_k$, then $F = \{D_1, \ldots, D_n\}$ attacks all the cells of I_1.That is I_1 is embedded and $\mathcal{A}$ is not a super partition of $\mathcal{P}$. That is $m = n$.
Moreover, if there exists a k such that $J_k = D_1D_2 \cdots D_s$ with $s > n$, then let F be a configuration on the $n \times n$ square containing C_k. The configuration $F' = F \setminus \{C_k\} \cup D_{n+1}$ is non-attacking and covers I_1, that is $\mathcal{A}$ is not a super partition of $\mathcal{P}$. The latter shows that $I_1, \ldots, I_n$ are the only intervals of the partition $\mathcal{A}$. By a similar argument, $J_1, J_2, \ldots, J_n$ are the only intervals of $\mathcal{B}$, that is $\mathcal{P}$ is the $n \times n$ square. □

Proposition 3.8. *Let $\mathcal{P}$ be a non-square polyomino with unique super partition $\mathcal{A}$. Then for any $J \in \mathcal{C} \setminus \mathcal{A}$, J is embedded.*

Proof. Given $J \in \mathcal{C} \setminus \mathcal{A}$ with $J = B_1B_2 \cdots B_m$, one can focus on the polyomino $\mathcal{P} = \mathcal{P}_J$ given by $I_1, \ldots, I_m \in \mathcal{A}$ such that for any $k \in 1, \ldots, m$ $I_k \cap J = B_k$. We proceed by induction on m. Let $m = 2$ and assume that $J = B_1B_2$ is not embedded. That is, if $D_1 \in I_1$ attacks B_1 and $D_2 \in I_2$ attacks B_2, D_1 and D_2 are attacking, hence there are no other cells in $\mathcal{P}$ except B_1, B_2, D_1, D_2 and $\mathcal{P}$ is the 2×2 square. Contradiction. Let $m > 2$. We assume that for any polyomino $\mathcal{P}'$ with super partition $\mathcal{B}$, any interval J' in $\mathcal{C}'_{\mathcal{P}} \setminus \mathcal{B}$ with $|J'| < m$ is embedded. We consider I_1. By the definition of polyomino $\mathcal{P}_J$, I_1 has a free edge interval, without loss of generality assume the super partition is made by horizontal intervals and the free edge is the uppermost one, and it contains the cell B_1 of J. Assume $I_1 = C_1C_2 \cdots C_n$ and $C_t = B_1$ for $1 < t \leq n$ (if $B_1 = C_1$, apply the following arguments to the cell C_n). Let $J_1 = D_1D_2 \cdots D_l$ with $D_1 = C_1$ and by construction $l \leq m$. We consider the polyomino $\mathcal{P}'$ given by $\mathcal{P} \setminus (I_1 \cup J_1)$. We divide two cases:

(1) $\mathcal{P}'$ is a square;
(2) $\mathcal{P}'$ is not a square.

In case (1), since $\mathcal{P}'$ contains $J \setminus B_1$, then $\mathcal{P}'$ is a $(m-1) \times (m-1)$ square and assume its cells are $\{A_{ij}\}_{i,j \in \{2,\dots,m\}}$. We prove that the length l of J_1 is equal to m and that C_2 and A_{22} are on the same column, by proving that $I_m = D_m A_{m2} A_{m3} \cdots A_{mm}$. By contraposition, assume $I_m = A_{m2} A_{m3} \cdots A_{mm}$ and a set $F \in \mathcal{R}_{\mathcal{P}'}$ with $|F| = m-1$ containing the cell $B_m \in I_m$. Then $F' = (F \setminus \{B_m\}) \cup \{B_1\} \in \mathcal{R}_{\mathcal{P}}$ since $B_1 \in I_1$. We have that F' attacks the interval $I_m \in \mathcal{A}$, that is $\mathcal{A}$ is not a super partition. This leads to a contradiction. In particular from $A_{m2} A_{m3} \cdots A_{mm} \subset I_m$ and $I_m \setminus J_1 = A_{m2} A_{m3} \cdots A_{mm}$, we obtain $I_m = D_m A_{m2} A_{m3} \cdots A_{mm}$. Moreover, by similar arguments, if the length n of I_1 is less than m, one can find an embedding for I_1, given by the set $\{D_2, A_{32}, \dots, A_{n+1n}\}$. That is $n \geq m$. If $n = m$, then $\mathcal{P}$ is the $m \times m$ square, that contradicts the hypothesis, hence $n > m$. That is, the $m \times m$ square is a subpolyomino of $\mathcal{P}$. Therefore, if $F \in \mathcal{R}_{\mathcal{P}}$ is a set of m non-attacking rooks on such a square containing B_1, that is $F \setminus B_1$ embeds $J \setminus B_1$, then $F' = (F \setminus \{B_1\} \cup \{C_n\})$ embeds the interval J.

In case (2), the set $\mathcal{A}' = \{I_2 \setminus \{D_2\}, \dots, I_l \setminus \{D_l\}, I_{l+1}, \dots, I_m\}$ is a partition of $\mathcal{P}'$. We prove that $\mathcal{A}'$ is a super partition. If one of the intervals $I_{l+1}, \dots, I_m$ is embedded, then it is embedded in $\mathcal{A}$, contradiction. If an interval $I_k \setminus D_k$ is embedded by a configuration $F' \in \mathcal{R}_{PP'}$, then $D_1 \cup F'$ embeds I_k in $\mathcal{P}$, and this is a contradiction. That is $\mathcal{A}'$ is a super partition of $\mathcal{P}'$. Moreover, the interval $J\ \{B_1\}$ has cardinality $m-1$ and by induction hypothesis is embedded by a configuration $F' \in \mathcal{R}_{\mathcal{P}'}$, and $F' \cup \{D_1\}$ is an embedding for J as desired. □

Lemma 3.9. *Let $\mathcal{P}$ be a polyomino and let $I, I' \in \mathcal{C}$ be such that I is embedded and there exists $J \in \mathcal{C}$ with $J \cap I \neq \varnothing$ and $J \cap I \neq \varnothing$. Then there exists an embedding F of I such that $F \cap I' \neq \varnothing$.*

Proof. Let $G = \{D_1, D_2, \dots, D_l\}$ be an embedding of $I = C_1 C_2 \cdots C_l$ and assume $G \cap I' = \varnothing$. Assume that $J \cap I = C_j$ and $J \cap I' = D$. Then we claim $F = G \setminus \{D_j\} \cup \{D\} \in \mathcal{R}_{\mathcal{P}}$. If D is attacked by a $D_k \in G$, then $D_k \in I'$, that is $G \cap I' \neq \varnothing$. Hence, F is an embedding for I with $F \cap I' \neq \varnothing$ as desired. □

We now prove the main theorem of this section

Theorem 3.10. *Let $\mathcal{P}$ be a polyomino. The following are equivalent:*

(1) $\mathcal{R}_{\mathcal{P}}$ is pure and has dimension $d-1$;
(2) $\mathcal{P}$ admits a super partition with $|\mathcal{A}| = d$.

Proof. (2) $\Rightarrow$ (1). Assume $\mathcal{P}$ has a super partition $\mathcal{A}$ with $|\mathcal{A}| = d$. By contraposition, assume that $\mathcal{R}_{\mathcal{P}}$ is not pure, that is there exists a facet F with $|F| = t < d$. Let $I_1, \dots, I_t \in \mathcal{A}$ be the intervals containing the t cells of F. Since $t < d$, then $\mathcal{A} \setminus \{I_1, \dots, I_t\} \neq \varnothing$, that is, there exists $I \in \mathcal{A} \setminus \{I_1, \dots, I_t\}$ that is embedded by F. Hence, $\mathcal{A}$ is not a super partition. Contradiction.

(1) $\Rightarrow$ (2). Assume that $\mathcal{R}_{\mathcal{P}}$ is pure and has dimension $d-1$. We divide the proof in the following steps:

- We prove that $\mathcal{P}$ admits a partition $\mathcal{A}$;
- If $\mathcal{E} \subseteq \mathcal{A}$ is the set of the embedded intervals of $\mathcal{A}$, then either $\mathcal{E} = \varnothing$ or $\mathcal{E} = \mathcal{A}$.
- In the case $\mathcal{A} = \mathcal{E}$, $\mathcal{P}$ admits a super partition $\mathcal{B}$.

Existence of $\mathcal{A}$. By contraposition, assume that $\mathcal{P}$ admits no partition, that is there exists a vertical interval I with a single cell C and a horizontal interval with a single cell D. Since $\mathcal{P}$ is a polyomino, the cells C and D are k-connected with k odd and let $C_1, C_2, \dots, C_k$ be the changes of direction. We consider the set $F = \{C = C_0, C_2, C_4 \dots, C_{k-1}, D = C_{k+1}\}$ that lies in $\mathcal{R}_{\mathcal{P}}$, because for any $i \in \{2, 4, \dots, k-1\}$ C_i only attacks C_{i-1} and C_{i+1}. Since $\mathcal{R}_{\mathcal{P}}$ is pure, then there exists $G \in \mathcal{R}_{\mathcal{P}}$ such that $|F \cup G| = d$. We now consider $F' = \{C_1, C_3, \dots, C_k\} \in \mathcal{R}_{\mathcal{P}}$ and we claim that $F' \cup G \in \mathcal{R}_{\mathcal{P}}$. If $A \in G$ attacks C_i with $i = 1, 3, \dots, k$, then A attacks either $C_{i-1} \in F$ or $C_{i+1} \in F$, that is $F \cup G \notin \mathcal{R}_{\mathcal{P}}$. Contradiction.
We prove that $F' \cup G$ is maximal, i.e. any cell A of $\mathcal{P}$ is attacked by a cell of $F' \cup G$. Since $F \cup G$ is maximal, then for any $A \in \mathcal{P}$ there exists $B \in F \cup G$ attacking A. If $B \in G$, then $B \in F' \cup G$. If $B \in F$, then $B = C_i$ for $i \in 0, 2, \dots, k+1$ and either C_{i-1} or C_{i+1} in F' attacks B, and hence A.

We showed that $F' \cup G$ with $|F' \cup G| < |F \cup G|$ is maximal, hence $\mathcal{R}_{\mathcal{P}}$ is not pure. Contradiction. Hence, $\mathcal{P}$ admits a partition $\mathcal{A}$ and since $\dim \mathcal{R}_{\mathcal{P}} = d-1$, then $|\mathcal{A}| \geq d$.

The set $\mathcal{E}$ of embedded intervals. Let $\varnothing \subseteq \mathcal{E} \subseteq \mathcal{A}$ be the set of embedded intervals of $\mathcal{A}$. We claim that either $\mathcal{E} = \varnothing$ or $\mathcal{E} = \mathcal{A}$. If $\varnothing \neq \mathcal{E} \neq \mathcal{A}$, let $I \in \mathcal{A} \setminus \mathcal{E}$ be a non-embedded interval and let $I' \in \mathcal{E}$, and since $\mathcal{A}$ is a partition, there exists $C \in I$ and $C' \in I'$ such that C and C' are $2k$-connected with $k \geq 1$ with intervals $I_1 = I, \dots, I_k = I' \in \mathcal{A}$. without loss of generality one can assume that $k = 1$, in fact if I_2 is embedded then take $I' = I_2$, otherwise take $I = I_2$. That is, assume I and I' are 2-connected, that is, there exists $J \in \mathcal{C}$ such that $J \cap I \neq \varnothing$ and $J \cap I' \neq \varnothing$. From Lemma 3.9, there exists an embedding F of I' such that $F \cap I \neq \varnothing$. Let G be a facet of $\mathcal{R}_{\mathcal{P}}$ containing F and let $G' = G \setminus \{A\} \cup C_j$. We observe that $G' \cap I = \varnothing$ that is either I is embedded or by Remark 3.4, there exists a facet $\bar{G}$ with $\bar{G} \cap I \neq \varnothing$ containing G', that are both contradictions. Hence either $\mathcal{E} = \varnothing$ or $\mathcal{E} = \mathcal{A}$. In the case $\mathcal{A} = \varnothing$, we have that $\mathcal{A}$ is a super partition and $|\mathcal{A}| = d$ due to Remark 3.4.

The case $\mathcal{E} = \mathcal{A}$. Any interval of $\mathcal{A}$ is embedded, that is $|\mathcal{A}| > d$ and, in particular, no interval of $\mathcal{A}$ has single cells, therefore $\mathcal{P}$ admits another partition $\mathcal{B}$. Assume that $\mathcal{A}$ contains rows and $\mathcal{B}$ contains columns. We claim that $\mathcal{B}$ contains no embedded intervals. By contraposition, assume that $J = C_1 C_2 \cdots C_l \in \mathcal{B}$ is embedded by $F = \{D_1, \dots, D_l\}$ and let $I_1, \dots, I_l$ be its rows. Since $l \leq d$ and $|\mathcal{A}| > d$, then there exists $I \in \mathcal{A}$ that is embedded by a facet G containing F. That is, a cell C_j of J is $2k$-connected to a cell D of I with $k \geq 1$ by a path with columns given by $J_1, J_2 \dots, J_k$ and changes of directions $L_1, L_2 \dots, L_{2k}$. We may assume that $k = 1$. In fact, if one of the rows of J_1 is embedded, then C_j is 2-connected to a cell D' of an embedded interval. Otherwise, from Lemma 3.9, one can choose F such that $J_1 \cap F = \{D_k\}$. Since G is maximal and J_1 has no embedded rows, any cell of $J_1 \setminus \{D_k\}$ is attacked by a cell of G, that is $G \setminus \{D_k\} \cup \{C_k\}$ is an embedding of J_1 and the cell $L_2 \in J_1$ is $2(k-1)$-connected to the cell D of I. That is, we assume $C_j \in J$ and $D \in I$ are 2-connected and we prove that there exists $\bar{F}$ embedding of I such that $F \cap \bar{F} \neq \varnothing$. For this aim, let F' be an embedding for I and assume $F' \cap F = \varnothing$. Let J' be such that $J' \cap I_j \neq \varnothing$ and $J' \cap I \neq \varnothing$. From Lemma 3.9 applied to J, J' and I_j,we can choose F such that $F \cap J' = \{D_j\}$. Moreover, since I is embedded, then $J' \cap F' = \{D'\}$. Take $\bar{F} = F' \setminus \{D'\} \cup \{D_j\}$, that is $F \cap \bar{F} \neq \varnothing$. Let G' be a maximal face containing $\bar{F} \cup F$, then a cell $A \in F \cap \bar{F}$ attacks a cell $C \in J$ and $D \in I$, that is the face $G' \setminus \{A\} \cup \{C, D\}$ is maximal and $|G' \setminus \{A\} \cup \{C, D\}| > G'$, contradiction to the pureness of $\mathcal{R}_{\mathcal{P}}$. This shows that $\mathcal{B}$ contains no embedded intervals, hence $|\mathcal{B}| = d$ and it is a super partition of $\mathcal{P}$ and by Remark 3.4 one has $|\mathcal{B}| = d$. □

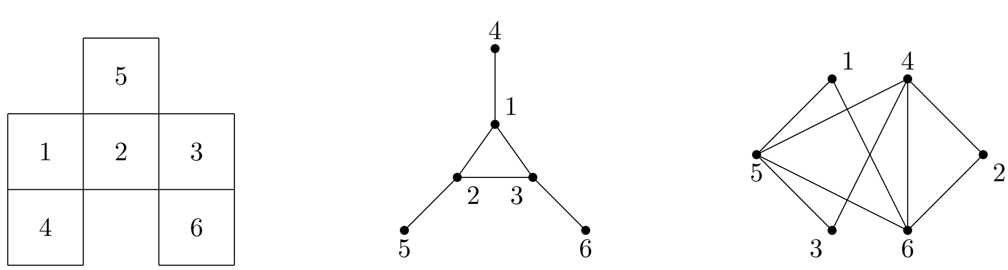
We recall the graph $G_{\mathcal{P}}$ is such that $V(G_{\mathcal{P}}) = \{C\}_{C \in \mathcal{P}}$ and

$$E(G_{\mathcal{P}}) = \{\{C, D\} : C \text{ and } D \text{ lie on the same row or column}\}.$$

Furthermore, the complement graph $\bar{G}_{\mathcal{P}}$ has edge set

$$E(\bar{G}_{\mathcal{P}}) = \{\{C, D\} : C \text{ and } D \text{ are non-attacking}\}.$$

An example of a polyomino $\mathcal{P}$ with the graphs $G_{\mathcal{P}}$ and $\bar{G}_{\mathcal{P}}$ can be found in Figure 4.



(A) $\mathcal{P}$ (B) $G_{\mathcal{P}}$ (C) $\bar{G}_{\mathcal{P}}$

FIGURE 4. A polyomino $\mathcal{P}$ and the graphs $G_{\mathcal{P}}$ and $\bar{G}_{\mathcal{P}}$

We observe that a partition on $\mathcal{P}$ induces a clique partition in the associated graph $G_{\mathcal{P}}$. Moreover, if such partition is *super* then the graph $G_{\mathcal{P}}$ is said to be *localizable* (see [8]). For a graph G, if $\Delta(G)$ is pure, then we say that G is *well-covered*. Hence, Theorem 3.10 can be rephrased as follows:

Theorem 3.11. *Let $G_{\mathcal{P}}$ be a graph associated to a polyomino $\mathcal{P}$. Then the following are equivalent:*

(1) $G_{\mathcal{P}}$ is well-covered;
(2) $G_{\mathcal{P}}$ is localizable.

4. Chordality of $\bar{G}_{\mathcal{P}}$

In this section we characterize the polyominoes $\mathcal{P}$ for which the graph $\bar{G}_{\mathcal{P}}$ is chordal. In view of Theorem 2.1, we obtain information on the minimal free resolution of $I_{\mathcal{R}}$. We recall that the *neighbourhood* of vertex $v \in V(G)$ of a graph G is defined as $N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$. We start with the following result.

Lemma 4.1. *Let $\mathcal{P}$ be a polyomino and let $\gamma = \{A_1, \dots, A_n\}$ with $n \geq 3$ be an induced cycle of $\bar{G}$. Then $n \in \{3, 4, 6\}$.*

Proof. We assume $n > 6$. Since γ is induced, then $A_3, A_4, A_5, A_6 \in N_G(A_1)$. Moreover, $\{A_3, A_5\}$, $\{A_4, A_6\} \in E(G)$, that is there exists a cell interval I of $\mathcal{P}$ containing A_1, A_3, A_5 and a cell interval J of $\mathcal{P}$ containing A_1, A_4, A_6, as shown in Figure 5. We have that $\{A_3, A_6\} \in E(\bar{G})$ and γ is not induced. Hence an induced cycle in $\bar{G}$ has length less than or equal to 6.

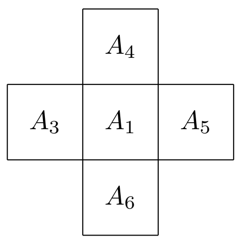


Figure 5

If $n = 5$, then $\{A_1, A_3\}, \{A_1, A_4\} \in E(G)$ and $\{A_3, A_4\} \notin E(G)$. That is there exists a cell interval I of $\mathcal{P}$ containing A_1 and A_3 and a cell interval J containing A_1 and A_4. By similar arguments, there exists a maximal cell interval $I' \neq J$ containing A_2 and A_4. Since $\{A_2, A_5\}, \{A_3, A_5\} \in E(G)$ and $\{A_2, A_4\} \notin E(G)$, then A_3 and A_2 are 1-connected with change of direction at A_5. This implies that either $\{A_1, A_5\}$ or $\{A_4, A_5\} \in E(G)$ and γ is not a cycle. This proves that $n \leq 6$ and $n \neq 5$ as desired. ☐

Remark 4.2. We highlight that a cycle $\gamma = \{A_1, \dots, A_6\}$ of length 6 is given by the hexomino in Figure 6. That is, whenever the above hexomino is a subpolyomino of $\mathcal{P}$ we have that $\bar{G}$ is not chordal.

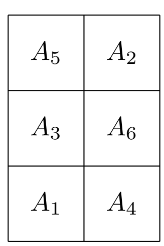


Figure 6

We prove some results under the assumption that $\bar{G}$ is chordal.

Remark 4.3. If $\bar{G}$ is chordal, then any two cells in $\mathcal{P}$ are k-connected with $k < 3$. In fact, assume that $C, D \in \mathcal{P}$ are k-connected with $C_1, \dots, C_k$ changes of directions and $k > 3$, that is C_1 and C_k lie on different cell intervals, hence $\{C, C_k, C_1, D\}$ is an induced 4-cycle of $\bar{G}$.

Proposition 4.4. *Let $\bar{G}$ be a chordal graph and assume that there exists $J \in \mathcal{C}$ such that $|J| > 2$. Then for any $I \in \mathcal{C} \setminus \{J\}$ one has $|I| = 2$.*

Proof. By contraposition, assume that there exists $I \in \mathcal{C} \setminus \{J\}$ such that $|I| > 2$. We distinguish two cases:

(1) $I \cap J \neq \varnothing$;
(2) $I \cap J = \varnothing$.

In case (1), let $I \cap J = \{C\}$. It follows that $|I \setminus \{C\}| \geq 2$ and $|J \setminus \{C\}| \geq 2$, that is there exist $C_1, C_2 \in I$ and $D_1, D_2 \in J$ such that $\{C_1, D_1, C_2, D_2\}$ is an induced 4-cycle of $\bar{G}$, that is $\bar{G}$ is not chordal.
In case (2), from Remark 4.3, a cell of I is at most 2-convex with a cell of J, that is let $I_1, \dots, I_l$ be intervals such that for any $j \in \{1, \dots, l\}$ $I_j \cap I \neq \varnothing$ and $I_j \cap J \neq \varnothing$. If $l > 2$, then the hexomino of Figure 6 is a subpolyomino of $\mathcal{P}$ and from Remark 4.2 $\bar{G}$ is not chordal. That is either $l = 1$ or $l = 2$. In the former case there exist $C_1, C_2 \in I \setminus (I_1 \cap I)$ and $D_1, D_2 \in J \setminus (I_1 \cap J)$ such that $\{C_1, D_1, C_2, D_2\}$ is an induced 4-cycle of $\bar{G}$, that is $\bar{G}$ is not chordal, while in the latter case there exist $C_1, C_2 \in I \setminus (I_1 \cap I)$ and $D_1, D_2 \in J \setminus (I_2 \cap J)$ such that $\{C_1, D_1, C_2, D_2\}$ is an induced 4-cycle of $\bar{G}$, that is $\bar{G}$ is not chordal. □

Corollary 4.5. *Let $\mathcal{P}$ be a polyomino with a chordal $\bar{G}$, then $\mathcal{P}$ is simple.*

Proof. In fact, if $\mathcal{P}$ is non-simple, then it is multiply connected, that is, there exists a closed path of cells. If such closed path has more than 4 intervals, then there are two cells that are k-connected with $k > 3$, that is $\bar{G}$ is not chordal due to Remark 4.3. That is there are 4 intervals $I_1, \dots, I_4$. Since $\mathcal{P}$ is non simple then we should have $|I_1| > 2$ and $|I_2| > 2$, that contradicts Proposition 4.4. □

For the aim of classifying the polyominoes having a chordal $\bar{G}$, we first characterize the simple non-thin ones.

Lemma 4.6. *Let $\mathcal{P}$ be a simple non-thin polyomino. Then $\bar{G}$ is chordal if and only if $\mathcal{P}$ is one of the two polyominoes in Figure 7.*

FIGURE 7

Proof. The graphs $\bar{G}$ associated to the polyominoes of Figure 7 are the graphs in Figure 8 that are clearly chordal.

FIGURE 8

We now assume that $\bar{G}$ is chordal and $\mathcal{P}$ strictly contains the square tetromino $\mathcal{Q} = \{C_1, C_2, C_3, C_4\}$ with $\{C_1, C_2\}$ non-attacking and let $C \in \mathcal{P} \setminus \mathcal{Q}$. Let I be the cell interval containing C. If there exists $D \in I$ such that $D \notin \{C_1, C_2, C_3, C_4\}$, then either $\{C, D, C_1, C_4\}$ or $\{C, D, C_1, C_3\}$ or $\{C, D, C_2, C_3\}$ or $\{C, D, C_2, C_4\}$ is an induced 4-cycle of $\bar{G}$. That is, if $I \setminus C \subseteq \{C_1, C_2, C_3, C_4\}$, assume $I = \{C, C_2, C_3\}$, as in Figure 9.

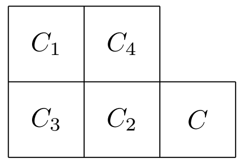
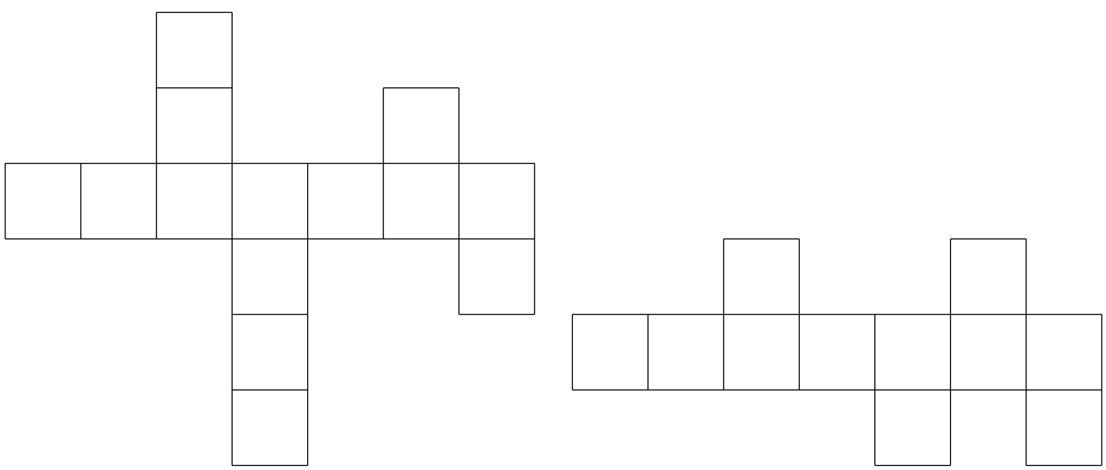


FIGURE 9

We assume that there exists $D \in \mathcal{P} \setminus \{C_1, C_2, C_3, C_4, C\}$ and let $J \in \mathcal{C}$ be such that $D \in J$. It follows that either $|J| > 3$ or $J = \{C, D\}$ and $J \cap \{C_1, C_4\} = \varnothing$. In the former case, both I and J have cardinality greater than 3, contradicting Proposition 4.4. In the latter case, C_1 and D are 3-connected contradicting Remark 4.3. That is, if $\bar{G}$ is chordal $\mathcal{P}$ is one of the two polyominoes in Figure 7. □

We are left with the characterization of the simple thin polyominoes having a chordal $\bar{G}$. We introduce a new class of polyominoes.

Definition 4.7. A simple thin polyomino $\mathcal{P}$ such that $\mathcal{C} = \{J, I_1, \ldots, I_l\}$ with $0 \leq l \leq |J|$ and for any $k \in \{1, \ldots, l\}$ $I_k \cap J \neq \varnothing$ is called a *brush polyomino*(see Figure 10A). If in addition for any $k \in \{1, \ldots, l\}$ $|I_k| = 2$, then is $\mathcal{P}$ is called a *short brush polyomino*(see Figure 10B).



(A) A brush polyomino (B) A short brush polyomino

Theorem 4.8. *Let $\mathcal{P}$ be a simple thin polyomino. Then $\bar{G}$ is chordal if and only if $\mathcal{P}$ is a short brush polyomino.*

Proof. Assume $\mathcal{P}$ is a brush polyomino and that $\{A_1, A_2, A_3, A_4\}$ is an induced cycle of $\bar{G}$ with A_1 and A_3 on a cell interval, and A_2 and A_4 on another cell interval. From the structure of $\mathcal{P}$ it follows that one of A_1 and A_3 lies on J, say A_3. Similarly, without loss of generality assume $A_2 \in J$, that is $A_2, A_3 \in E(G)$ and $\bar{G}$ is not a cycle.
We now assume that $\mathcal{P}$ is a simple thin polyomino with a chordal $\bar{G}$. From Remark 4.3, we have that $\mathcal{P}$ is k-connected with $k \leq 2$. We now distinguish two cases.

(1) for any $I \in \mathcal{C}$ it holds $|I| = 2$;

(2) there exists $J \in \mathcal{C}$ such that $|J| > 2$.

In case (1), we have that $|\mathcal{C}| \leq 3$, in fact the only polyominoes satisfying the above property are subpolyominoes of the skew tetromino in Figure 11. In fact, if $\mathcal{C}$ contains another cell interval I with $|I| = 2$, then I contains C (resp. D) and another cell A that is 3-connected to D (resp. C).

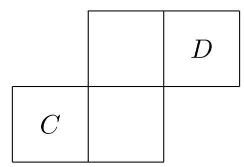


FIGURE 11. The skew tetromino

In case (2), from Proposition 4.4, we obtain that $\forall I \in \mathcal{C} \setminus \{J\}$ one has $|I| = 2$. We are left with proving that $\forall I \in \mathcal{C} \setminus \{J\}$ we have $I \cap J \neq \varnothing$. Assume $I \cap J = \varnothing$. Since $I = \{C_1 C_2\}$ and $\mathcal{P}$ is simple thin, then there exists $I_1 \in \mathcal{C}$ with $|I_1| = 2$ such that $I_1 = \{C_2 C\}$ for some cell $C \in J$. Since $|J \setminus \{C\}| \geq 2$, then there exists $D_1, D_2 \in J$ such that $\{C_1, D_1, C_2, D_2\}$ is an induced 4-cycle of $\bar{G}$, that is $\bar{G}$ is not chordal. This leads to a contradiction. It follows that $\mathcal{P}$ is a brush polyomino. $\square$

From Theorem 2.1 one obtains the following

Corollary 4.9. *Let $\mathcal{P}$ be a simple polyomino. Then $\operatorname{reg} R/I(G_{\mathcal{P}}) \leq 1$ if and only if $\mathcal{P}$ is a brush polyomino or $\mathcal{P}$ is one of the polyominoes in Figure 7.*

5. The Castelnuovo-Mumford regularity of pure brush polyominoes

In this section, we compute the Castelnuovo-Mumford regularity of $R/I(G_{\mathcal{P}})$ for the brush polyominoes $\mathcal{P}$ that have a pure rook complex. For this, we call them *pure brush polyominoes*. We are motivated by the following observations.

Remark 5.1. Let $\mathcal{P}$ be a simple thin polyomino. We observe that $G_{\mathcal{P}}$ contains no induced cycles, hence is chordal and by [18, Corollary 7.(2)] $\mathcal{R}_{\mathcal{P}}$ is vertex decomposable.

Corollary 5.2. *Let $\mathcal{P}$ be a simple thin polyomino. Then the following are equivalent*

(i) $\mathcal{R}_{\mathcal{P}}$ is pure;
(ii) $\mathcal{R}_{\mathcal{P}}$ is Cohen-Macaulay;
(iii) $\mathcal{R}_{\mathcal{P}}$ is vertex decomposable.

Therefore, if $\mathcal{R}_{\mathcal{P}}$ is pure and $h(\mathcal{R}_{\mathcal{P}}) = (h_0, \ldots, h_r)$, then $r = \operatorname{reg} R/I(G)$. Hence, for the rest of the section we focus on the class of pure brush polyominoes (see Figure 12).

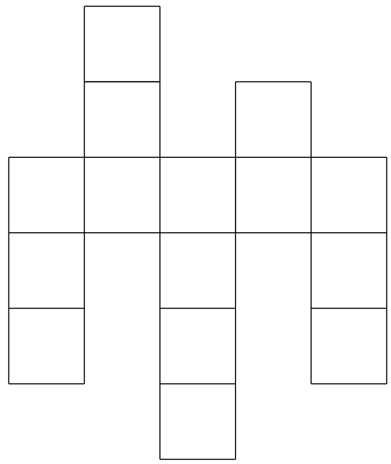


FIGURE 12. A pure brush polyomino

If follows that a pure brush polyomino with $\dim \mathcal{R}_{\mathcal{P}} = d-1$ is such that $\mathcal{A} = \{I_1, \dots, I_d\}$ with $|I_k| = \ell_k$ for any $k \in \{1, \dots, d\}$, $\mathcal{C} = \mathcal{A} \cup \{J\}$ with $|J| = d$, and $I_k \cap J \neq \varnothing$ for any $k \in \{1, \dots, d\}$. Let $\ell = (\ell_1, \dots, \ell_d)$. We want to study the vectors $f(\mathcal{R}_{\mathcal{P}})$ and $h(\mathcal{R}_{\mathcal{P}})$ for a pure brush polyomino $\mathcal{P}$.

We recall that for $1 \leq k \leq d$ the k-th elementary symmetric polynomial in d indeterminates $X_1, \dots, X_d$ is

$$\varepsilon_k^{(d)}(X_1, \dots, X_d) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq d} X_{i_1} X_{i_2} \cdots X_{i_k}.$$

For any $1 \leq k \leq d$ we set

$$\begin{aligned} \sigma_k &= \varepsilon_k(\ell_1, \dots, \ell_d), \\ \sigma'_k &= \varepsilon_k(\ell_1 - 1, \dots, \ell_d - 1), \\ \sigma''_k &= \varepsilon_k(\ell_1 - 2, \dots, \ell_d - 2). \end{aligned}$$

Lemma 5.3. *For any $d \in \mathbb{N}$ and $1 \leq k \leq d$ the following relations hold*

(i) $\sigma'_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} \sigma_i$;

(ii) $\sigma_k = \sum_{i=0}^{k} \binom{d-i}{k-i} \sigma'_i$;

(iii) $\sigma''_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} \sigma'_i$.

Proof. We consider a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, d\}$ and we consider the number

$$(\ell_{i_1} - 1)(\ell_{i_2} - 1) \cdots (\ell_{i_k} - 1) \tag{3}$$

For any $i \in \{1, \dots, k\}$ we set

$$\sigma_i^{(k)} = \varepsilon_i^{(k)}(\ell_{i_1}, \dots, \ell_{i_k}),$$

hence, from Vieta's formulas, Equation (3) becomes

$$\sum_{i=0}^{k} (-1)^{k-i} \sigma_i^{(k)}.$$

We now consider

$$\sigma'_k = \sum_{1\leq i_1<i_2<\ldots<i_k\leq d} (\ell_{i_1}-1)(\ell_{i_2}-1)\cdots(\ell_{i_k}-1) = \sum_{1\leq i_1<i_2<\ldots<i_k\leq d} \sum_{i=0}^{k}(-1)^{k-i}\sigma_i^{(k)}.$$

We observe that fixed $\{i_1,\ldots,i_k\}$ and $A\subseteq\{i_1,\ldots,i_k\}$ with $|A|=i$, the summand

$$\prod_{a\in A}\ell_a$$

appears $\binom{d-j}{k-j}$ times in σ'_k. That is

$$\sigma'_k = \sum_{1\leq i_1<i_2<\ldots<i_k\leq d} \sum_{i=0}^{k}(-1)^{k-i}\sigma_i^{(k)} = \sum_{i=0}^{k}(-1)^{k-i}\binom{d-j}{k-j}\sigma_k,$$

and relation (i) follows. Similarly, relation (iii) follows.

Relation (ii) follows from Relation (i), as a comparision with the relation between Equation 1 and Equation 2. □

Moreover we have

Theorem 5.4. *Let $\mathcal{P}$ be a pure brush polyomino with $\dim\mathcal{R}_{\mathcal{P}}=d-1$. Then the following relations hold*

(1) for all $k\in\{1,\ldots,d\}$

$$f_{k-1}=\sigma'_k+(d-(k-1))\sigma'_{k-1};$$

(2) for all $t\in\{0,\ldots,d\}$

$$h_t=\sigma''_t+(d-(t-1))\sigma''_{t-1}.$$

Proof. We prove relation (1), by first observing that for any $i=2,\ldots,d$ we have

$$f_{k-1}=\sigma_k-\left(\sum_{j=0}^{k-2}\binom{d-j}{k-j}\sigma'_j\right).$$

We have that for any $\{i_1,\ldots,i_k\}\subset\{1,\ldots,d\}$, we have

$$\ell_{i_1}\cdots\ell_{i_k}$$

configurations of rooks. From this number for any $0\leq j\leq k-2$ we have to subtract the configurations that have exactly $k-j$ cells on the common interval J. Fixed a subset $A=\{a_1,\ldots,a_{k-j}\}\subset\{i_1,\ldots,i_k\}$, the configurations containing the cells in $J\cap\{I_{a_1}\},\ldots,J\cap\{I_{a_{k-j}}\}$ are

$$\prod_{t\in\{i_1,\ldots,i_k\}\setminus A}(\ell_t-1).$$

Let $\mathcal{S}_{k-j}$ be the set of cardinality $k-j$ subsets of $\{i_1,\ldots,i_k\}$. We consider

$$\sum_{1\leq i_1<i_2<\ldots<i_k\leq d}\ \sum_{A\in\mathcal{S}_{k-j}}\ \prod_{t\in\{i_1,\ldots,i_k\}\setminus A}(\ell_t-1).$$

Since the j elements of $\{i_1,\ldots,i_k\}\setminus A$ are fixed, then in the above sum any product $\binom{d-j}{k-j}$ is counted times, and their sum retrieves σ'_j. Hence,

$$f_{k-1}=\sum_{1\leq i_1<i_2<\ldots<i_k\leq d}\ell_{i_1}\cdots\ell_{i_k}-\left(\sum_{j=0}^{k-2}\sum_{A\in\mathcal{S}_{k-j}}\prod_{t\in\{i_1,\ldots,i_k\}\setminus A}(\ell_t-1)\right)=$$

$$=\sum_{1\leq i_1<i_2<\ldots<i_k\leq d}\ell_{i_1}\cdots\ell_{i_k}-\left(\sum_{j=0}^{k-2}\ \sum_{1\leq i_1<i_2<\ldots<i_k\leq d}\ \sum_{A\in\mathcal{S}_{k-j}}\ \prod_{t\in\{i_1,\ldots,i_k\}\setminus A}(\ell_t-1)\right)=$$

$$= \sigma_k - \sum_{j=0}^{k-2} \binom{d-j}{k-j} \sigma'_j.$$

According to relation (ii) of Lemma 5.3, we have that

$$f_{k-1} = \sum_{j=k-1}^{k} \binom{d-j}{k-j} \sigma'_j = \sigma'_k + (d-(k-1))\sigma'_{k-1}.$$

To prove relation (2), we consider Equation (1), that is

$$h_t = \sum_{k=0}^{t} (-1)^{t-k} \binom{d-k}{t-k} f_{k-1} =$$

$$= (-1)^t \binom{d}{t} f_{-1} + \sum_{k=1}^{t} (-1)^{t-k} \binom{d-k}{t-k} f_{k-1}.$$

By using relation (1), we obtain

$$= (-1)^t \binom{d}{t} f_{-1} + \sum_{k=1}^{t} (-1)^{t-k} \binom{d-k}{t-k} (\sigma'_k + (d-(k-1))\sigma'_{k-1}) =$$

one observes that $f_{-1} = \sigma'_0 = 1$

$$= \sum_{k=0}^{t} (-1)^{t-k} \binom{d-k}{t-k} \sigma'_k + \sum_{k=1}^{t} (-1)^{t-k} \binom{d-k}{t-k} (d-k+1)\sigma'_{k-1} = (*).$$

We observe that

$$(d-k+1)\binom{d-k}{t-k} = (d-k+1)\frac{(d-k)!}{(t-k)!(d-t)!} \cdot \frac{d-t+1}{d-t+1} = (d-t+1)\binom{d-k+1}{t-k},$$

that is

$$(*) = \sum_{k=0}^{t} (-1)^{t-k} \binom{d-k}{t-k} \sigma'_k + (d-t+1) \sum_{k=1}^{t} (-1)^{t-k} \binom{d-k+1}{t-k} \sigma'_{k-1} =$$

In the second sum, we substitue $j = k-1$ and we obtain

$$= \sum_{k=0}^{t} (-1)^{t-k} \binom{d-k}{t-k} \sigma'_k + (d-t+1) \sum_{j=0}^{t-1} (-1)^{t-j+1} \binom{d-j}{t-1-j} \sigma'_j.$$

By using relation (iii) of Lemma 5.3, we obtain that

$$h_t = \sigma''_t + (d-t+1)\sigma''_{t-1},$$

as desired. □

We reformulate the definition of induced matching number for the graph $G_{\mathcal{P}}$, denoted by $\nu(G_{\mathcal{P}})$, in terms of the intervals of $\mathcal{P}$.

Definition 5.5. An induced matching is a set of edges $\{\{A_1, B_1\}, \dots, \{A_n, B_n\}\}$ such that for any $j, k \in \{1, \dots, n\}$, if $\{A_j, B_j\} \subset I_j \in \mathcal{C}$ and $\{A_k, B_k\} \subset I_k \in \mathcal{C}$, then there is no $J \in \mathcal{C}$ such that

$$J \cap I_j \subset \{A_j, B_j\} \text{ and } J \cap I_k \subset \{A_k, B_k\}.$$

For a polyomino $\mathcal{P}$, we set $\mathcal{S} = \{I \in \mathcal{C} : I \text{ has at least 2 single cells}\}$.

Lemma 5.6. *Let $\mathcal{P}$ be a simple polyomino. Then*

$$\nu(G_{\mathcal{P}}) \geq |\mathcal{S}|.$$

Proof. Let $\mathcal{S} = \{I_1, \dots, I_m\}$. For any $j \in \{1, \dots, m\}$, let A_j and B_j be two single cells of I_j. It follows that

$$\{\{A_1, B_1\}, \dots, \{A_m, B_m\}\}$$

is an induced matching, hence $\nu(G_{\mathcal{P}}) \geq |\mathcal{S}|$. $\square$

Corollary 5.7. *Let $\mathcal{P}$ be a pure brush polyomino. Then*

$$\operatorname{reg} R/I(G_{\mathcal{P}}) = \nu(G_{\mathcal{P}}).$$

Proof. Let $\mathcal{P}$ be a pure brush polyomino with $\mathcal{C} = \{J, I_1, \dots, I_d\}$. We recall that $\ell_k = |I_k|$ for any $k \in \{1, \dots, d\}$. We observe that if $l_k \geq 3$ then I_k has two single cell. We distinguish two cases

(1) for any $k \in \{1, \dots, d\}$ $\ell_k \geq 3$;
(2) there exists $j \in \{1, \dots, d\}$ such that $\ell_j = 2$;

In case (1), from Lemma 5.6 we have that $d = \nu(G_{\mathcal{P}})$, that is from Theorem 2.2

$$\nu(G_{\mathcal{P}}) \leq \operatorname{reg} R/I(G_{\mathcal{P}}) \leq d,$$

and the assertion follows.
In case (2), we relabel the intervals $I_1, \dots, I_d$ in a way such that

$$\ell_1, \dots, \ell_t > 3 \text{ and } \ell_{t+1} = \ldots = \ell_d = 2$$

for $t < d$. In this case we have $\nu(G_{\mathcal{P}}) = t + 1$. In fact, since $\mathcal{S} = \{I_1, \dots, I_t\}$, then we have t edges in an induced matching. To this we add the unique edge arising from I_{t+1}. Combining the facts $I_k \cap J \neq \varnothing$ for any $k \in \{1, \dots, d\}$ and $\ell_{t+1} = \ldots = \ell_d = 2$, we obtain that the intervals $I_1, \dots, I_{t+1}$ give rise to an induced matching. We prove that $h_k = 0$ for $k > t + 1$.

Let $k > t + 1$. In any cardinality-k subset of $\ell_1 - 2, \dots, \ell_d - 2$ there is a 0, that is

$$\sigma''_k = \sigma''_{k-1} = 0.$$

From relation (2) of Theorem 5.4, we obtain $h_k = 0$ as desired. $\square$

As a conclusion, in this paper we characterize the polyominoes having a pure rook complex. It could be of interest finding a characterization for those polyominoes having a Cohen-Macaulay (shellable, vertex decomposable) rook complex, and among them finding the Gorenstein ones.

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