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Freely vibrating nanoplates on nanofoundations

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ABSTRACT

Free vibrations of Kirchhoff axisymmetric nanoplates resting on elastic nano-foundations are investigated accounting for long range interactions through nonlocal methodologies. Size effects are captured by means of the stress-driven integral elasticity. The interaction between plate and supporting medium is modeled by a nonlocal theory according to which the foundation reaction is spatial convolution driven by the displacement field. This formulation represents an advanced methodology with respect to the Winkler (local) approach, leading to a well-posed structure-foundation problem. The governing equation of motion for freely vibrating nanoplates resting on nanofoundations are derived and expressed in nondimensional form, leading to a high-order nonautonomous differential problem. Natural frequencies and mode shapes are computed numerically exploiting the Compound Matrix Method. Case studies involving nanoplates with simply supported and clamped edges are presented to highlight the influence of the characteristic length parameters on the dynamic response. The obtained outcomes show the significant role played by nonlocal effects in accurately predicting the vibrational behavior of nanoplates interacting with elastic substrates.

1. Introduction

Smart small-scale systems have recently gained considerable interest due to their remarkable features, such as rapid and highly sensitive responses, reduced power demand and production costs, and straightforward integration into compact electronic devices. Within this framework, nanoplates emerge as particularly promising structural components for small-scale applications (see e.g. Eroğlu et al. (2025), Li et al. (2025), Momeni-Khabisi and Tahani (2026), Wang et al. (2025) and Yıldırım and Esen (2025)).

It is widely recognized that the design and optimization of innovative materials and advanced structural systems call for refined methodologies able to describe unconventional mechanical phenomena (Barchiesi et al., 2021; dell'Isola et al., 2016; Hozhabrossadati et al., 2018; Izadi et al., 2024; Nguyen et al., 2025; Özdemir et al., 2024; Penna & Lovisi, 2025; Placidi et al., 2024, 2016, 2022; Rezaei et al., 2025). Indeed, accurate modeling of structures at the micro- and nano-scale requires sophisticated approaches capable of capturing scale effects that are negligible at the macroscale. Nonlocal continuum mechanics has therefore become a valuable tool for describing mechanical behavior of nanostructures, as it explicitly incorporates size dependent behaviors into the constitutive response.

The groundwork of nonlocal continuum mechanics was laid in early pioneering contributions by Kröner (1967), Krumhansl (1968), Kunin (1968) and Rogula (1965, 1982). Subsequently, Eringen (1972, 1983) developed an integral formulation of nonlocal

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elasticity, in which the stress at a material point is expressed by a spatial convolution of the elastic strain field, thus accounting for long-range interactions. Eringen's differential constitutive law was subsequently applied by Peddieson et al. (2003) to investigate size effects in nanobeams, leading to paradoxical outcomes. The issue was further investigated by Benvenuti and Simone (2013), Fernández-Sáez et al. (2016) and Khodabakhshi and Reddy (2015) and ultimately solved by Romano, Barretta, Diaco and de Sciarra (2017), clarifying that the Eringen's nonlocal integral can be equivalently reverted into a differential formulation only by satisfying homogeneous boundary conditions of constitutive type. As proven by Romano, Barretta, Diaco and de Sciarra (2017), ill-posedness of the Eringen-based structural problems stems from an intrinsic incompatibility between equilibrium conditions and constitutive law. These shortcomings were entirely bypassed by the stress-driven nonlocal integral model proposed by Romano and Barretta (2017). According to this formulation, the nonlocal elastic strain is a spatial convolution driven by the stress field by means of a proper averaging kernel. This approach guaranties the well-posedness of the governing structural problem while effectively capturing long-range interactions. The stress-driven nonlocal model has been then successfully applied in several subsequent investigations by Behnam-Rasouli et al. (2024), Caporale et al. (2023), Indronil (2025), Jafarinezhad et al. (2023a), Lovisi (2023), Lovisi et al. (2024), Rezaiee-Pajand and Rajabzadeh-Safaei (2022), Wang et al. (2023) and Zhang et al. (2022). A broad review of recent advancements in the nonlocal nanostructural modeling is presented by Barretta et al. (2023).

Exploiting the stress-driven approach, elastostatics of axisymmetric nanoplates has been investigated by Barretta et al. (2019) and Vaccaro and Sedighi (2023), where the governing problem is converted into a set of differential equations equipped with constitutive boundary conditions. Functionally graded annular nanoplates have been investigated by Jafarinezhad et al. (2023b), based on the Helmholtz attenuation function, and by Cianci et al. (2025), assuming a Gaussian kernel. Axisymmetric thick plates have been investigated by Jafarinezhad et al. (2024). Notably, analytical solutions of the stress-driven nonlocal elastic equilibrium problem of axisymmetric plates have been provided by Cianci et al. (2024) and Jafarinezhad et al. (2023b) proving that the nonlocal theory proposed by Barretta et al. (2019) leads to well-posed plate problems.

A further challenging aspect concerns modeling of the structure-foundation interaction (see e.g. Alnujaie et al. (2024), Chu et al. (2026) and Daikh et al. (2023)). Indeed, considerable attention has to be devoted to the investigation of nanoplates resting on elastic substrates, as they are essential for modeling support conditions in nano-electromechanical systems. Accurate modeling of scale effects in nanofoundations is thus needed. The simplest model of elastic medium was introduced by Winkler (1867), who proposed a purely local constitutive law in which the foundation reaction is proportional to the displacement field (of the structure-soil interface surface) through an elastic stiffness, neglecting long-range interactions. A crucial extension was later proposed by Wieghardt (1922), incorporating nonlocal effects through an integral formulation, where the displacement field is obtained by spatial convolution driven by the foundation reaction field. However, Wieghardt's theory was shown to be ill-posed due to an incompatibility between constitutive and kinematic equations, as observed by Sollazzo (1966) and van Langendonck (1962) and by Wieghardt himself. To overcome this issue, fictitious boundary forces were introduced by Sollazzo (1966), yielding a mathematically consistent but physically questionable model. Alternative modeling strategies were developed within differential approaches. Notably, Pasternak (1954) generalized Winkler's model by including shearing interactions among the foundation springs, while Reissner (1958) further extended this theory by introducing an additional internal length scale. A generalization of Wieghardt (1922) foundation for two-dimensional continua was provided by Capurso (1967). Nevertheless, a gap still remained regarding consistency of nonlocal integral theory of foundations. To address this need, an integral nonlocal model of elastic media has been proposed by Vaccaro et al. (2021) to consistently capture long range interactions in elastic media. According to this formulation, the foundation reaction field is a spatial convolution driven by the local (Winkler) response. This approach is here extended to plates resting on nanofoundations.

2. Motivation and outline

By extending the contribution provided by Barretta et al. (2022) to two-dimensional nanostructures, the present work investigates elastodynamics of nanoplates interacting with nanofoundations. A consistent nonlocal methodology is proposed, involving a minimum number of parameters (i.e. plate and foundation length scale parameters), while effectively capturing both stiffening and softening structural responses.

In modeling of nanosystems, both the internal elasticity of the nanostructure and the external elasticity of the surrounding foundation can be accurately described through nonlocal continuum models, provided that the coupled structure–foundation problem is mathematically well-posed. Ensuring such well-posedness remains an open issue in Engineering Science. Indeed, the formulation of internal nonlocal elasticity proposed by Eringen (1972) leads to ill-posed structural problems due to an incompatibility with the equilibrium requirements (Romano, Barretta and Diaco, 2017). The nonlocal formulation proposed by Wieghardt (1922) to capture long range interactions in elastic foundations does not guaranty well-posedness of the relevant continuum problem due to a conflict with kinematic compatibility (Vaccaro et al., 2021).

The present work addresses this challenging issue by proposing an alternative nonlocal formulation to the Eringen–Wieghardt integral elasticity, leading to well-posed nonlocal problems of nanoplates supported by nano-scale foundations. The displacement-driven integral approach (Vaccaro et al., 2021) adopted by Barretta et al. (2022) to investigate elastostatics of nanosystems, exploited by Das et al. (2025) to assess dynamics of nanobeams on nanofoundations and recently applied by Simões and Pinto da Costa (2026) to investigate dynamics of a beam on a nonlocal medium under a moving loading, is here extended to nanoscale plate-foundation dynamical problem. This work thus enables the modeling of nanosystems in which accounting for the structure-substrate interaction is crucial for design of nanosystems in point-of-care devices, wearable electronics and tissue engineering applications (see e.g. Bai et al. (2019), Batool et al. (2025), Chen et al. (2023) and Kong et al. (2024)).

Here is the outline. In Section 3, kinematics and equilibrium of Kirchhoff axisymmetric annular plates is introduced. Then, nonlocal modeling of nanoplates resting on nanofoundations is provided in Section 4. In Section 5, the governing equations of motion along with standard and non-standard boundary conditions are derived. The free vibration nonlocal problem is formulated and put into a nondimensional form. Numerical analyses are finally provided in Section 6, highlighting the effects of plate and foundation length scale parameters on natural frequencies and mode shapes. Concluding remarks are discussed in Section 7.

3. Kirchhoff axisymmetric annular plates

A cylindrical coordinate system r, θ, z is adopted to model a thin axisymmetric plate of uniform thickness h, with internal and external radii r_i and r_e, respectively. The plate planform occupies the domain $\Omega = [r_i, r_e] \times [0, 2\pi]$ in the $r\theta$ plane, where the base unit vectors are the radial vector $\mathbf{e}_r$ and the circumferential vector $\mathbf{e}_\theta$. The edges at $r = r_i$ and $r = r_e$ are denoted by $\partial\Omega_i$ and $\partial\Omega_e$. Denoting with ∇ the gradient operator, the bending curvature tensor, according to the (linearized) Kirchhoff plate theory, is defined by $\chi := \nabla\nabla u$, where $u : [r_i, r_e] \to \mathbb{R}$ is the transverse displacement field. Exploiting the radial derivative ∂_r and the tensor product $\otimes$, the tangent curvature tensor is expressed as follows

$$\chi = \chi_r \, \mathbf{e}_r \otimes \mathbf{e}_r + \chi_\theta \, \mathbf{e}_\theta \otimes \mathbf{e}_\theta \,, \tag{1}$$

where $\chi_r := \partial_r^2 u$ is the radial curvature and $\chi_\theta := \dfrac{\partial_r u}{r}$ the circumferential one.

The stress tensor is given by $\mathbf{M} = M_r \, \mathbf{e}_r \otimes \mathbf{e}_r + M_\theta \, \mathbf{e}_\theta \otimes \mathbf{e}_\theta$, where M_r and M_θ are the radial and circumferential bending interaction fields. The principle of virtual work states that the internal virtual work $\int_\Omega \mathbf{M} : \chi_{\delta u} \, dA$ equals the external virtual work for all admissible virtual displacements δu, with $\chi_{\delta u}$ denoting the curvature tensor field associated with δu. After integration by parts and localization, the equilibrium differential equation takes the form

$$\frac{1}{r}\left(\partial_r^2(M_r \, r) - \partial_r M_\theta\right) = q \,, \quad r \in \Omega \,, \tag{2}$$

subject to the boundary conditions

$$\begin{cases} M_r \, \partial_r \delta u = -\bar{M}_i \, \partial_r \delta u \,, & r \in \partial\Omega_i \,, \\ M_r \, \partial_r \delta u = \bar{M}_e \, \partial_r \delta u \,, & r \in \partial\Omega_e \,, \\ \left(M_\theta - \partial_r(M_r \, r)\right) \delta u = -\bar{Q}_i \, r \, \delta u \,, & r \in \partial\Omega_i \,, \\ \left(M_\theta - \partial_r(M_r \, r)\right) \delta u = \bar{Q}_e \, r \, \delta u \,, & r \in \partial\Omega_e \,. \end{cases} \tag{3}$$

The shearing interaction field is defined by

$$Q_r := \frac{M_\theta - \partial_r(M_r, r)}{r} \,. \tag{4}$$

A circular plate is obtained as the limiting case of an annular plate with a free internal edge for a vanishing internal radius, i.e. $r_i \to 0$.

4. Stress-driven nonlocal plates on displacement-driven nonlocal foundations

For a local isotropic elastic plate, the elastic curvature fields are related to the bending interaction fields by

$$\begin{vmatrix} \chi_r^{el} \\ \chi_\theta^{el} \end{vmatrix} = \frac{1}{D\,(1-\nu^2)} \begin{bmatrix} 1 & -\nu \\ -\nu & 1 \end{bmatrix} \begin{vmatrix} M_r \\ M_\theta \end{vmatrix} \tag{5}$$

where ν is the Poisson's ratio and $D := \dfrac{E\,h^3}{12\,(1-\nu^2)}$ is the flexural stiffness. Since no inelastic effects are present, the superscript el will be omitted in the following.

To include size-dependent behaviors, the stress-driven nonlocal model of elasticity for axisymmetric annular nanoplates is exploited. By virtue of the axisymmetry, the nonlocal elastic radial curvature is expressed as spatial convolution of the local one, i.e.

$$\chi_r(r) = \int_{r_i}^{r_e} \phi_{l_c}(r-\xi) \, \frac{M_r(\xi) - \nu M_\theta(\xi)}{D\,(1-\nu^2)} \, d\xi \tag{6}$$

where ϕ_{l_c} is the averaging kernel depending on the plate characteristic length l_c. As proposed by Barretta et al. (2019), such a constitutive choice is consistent with the axisymmetry of the structural problem. The same physically sound methodology is exploited here and extended to dynamics of nanoplates interacting with nanofoundations.

To enable explicit inversion of Eq. (6), the bi-exponential Helmholtz kernel is chosen:

$$\phi_{l_c}(r) = \frac{1}{2\,l_c} \, \exp\left(-\frac{|r|}{l_c}\right) , \tag{7}$$

which satisfies the properties of positivity, symmetry, normalization and limit impulsivity on the real axis. As proven by Barretta et al. (2019), the integral Eq. (7) is equivalent to the differential constitutive law

$$\frac{\chi_r}{l_c^2} - \partial_r^2 \chi_r = \frac{M_r - \nu M_\theta}{l_c^2\, D\,(1-\nu^2)}\,, \tag{8}$$

with constitutive boundary conditions

$$\begin{cases} \partial_r \chi_r(r_i) - \dfrac{1}{l_c}\, \chi_r(r_i) = 0\,, \\ \partial_r \chi_r(r_e) + \dfrac{1}{l_c}\, \chi_r(r_e) = 0\,. \end{cases} \tag{9}$$

Thanks to the choice of the Helmholtz kernel, the nonlocal constitutive law can reverted into a differential formulation, avoiding integral operators.

The foundation reaction is modeled according to the nonlocal displacement-driven approach adopted by Vaccaro et al. (2021). This approach is here extended to plates interacting with nonlocal elastic media.

The reaction field f is expressed by the spatial convolution between the Winkler-type response $k_W\, u$ and the averaging kernel ϕ_{c_f}, i.e.

$$f(r) = \int_{r_i}^{r_e} \phi_{c_f}(r-\xi)\, k_W\, u(\xi)\, d\xi\,, \tag{10}$$

where ϕ_{c_f} is described by the foundation characteristic length c_f. The integration range has been limited to the structural domain $[r_i, r_e]$ since it is the range within the nanobeam interacts with the nanofoundation. Indeed, the displacement field driving the spatial convolution in Eq. (10) denotes the transverse displacement of the interface between the structure and the elastic medium.

Again, choosing the Helmholtz kernel the following equivalent differential form can be apoted, i.e.

$$\frac{f}{c_f^2} - \partial_r^2 f = \frac{k_W}{c_f^2} u\,, \tag{11}$$

equipped with constitutive boundary conditions

$$\begin{cases} \partial_r f(r_i) - \dfrac{1}{c_f} f(r_i) = 0\,, \\ \partial_r f(r_e) + \dfrac{1}{c_f} f(r_e) = 0\,. \end{cases} \tag{12}$$

As shown by Vaccaro et al. (2021), the constitutive boundary conditions in Eq. (12) along with the differential equation (11) assure that the constitutive differential problem is equivalent to the integral constitutive law in Eq. (10).

It is worth noting that the spatial convolution in Eq. (10) provides the reaction field $f(r)$ for $r \in [r_i, r_e]$, which corresponds to the structural domain of interest. Nevertheless, from a mathematical point of view, the field f could not vanish for $r < r_i$ and $r > r_e$, even if the displacement field u is assumed to vanish over the considered range. In particular, its functional expression is given as follows:

$$\begin{cases} f(r) := \displaystyle\int_{-\infty}^{+\infty} \phi_{c_f}(r,\xi)\, u(\xi)\, d\xi = \int_{r_i}^{r_e} \frac{e^{\frac{r-\xi}{c_f}}}{2c_f}\, u(\xi)\, d\xi, \qquad r \le r_i\,, \\ f(r) := \displaystyle\int_{-\infty}^{+\infty} \phi_{c_f}(r,\xi)\, u(\xi)\, d\xi = \int_{r_i}^{r_e} \frac{e^{-\frac{r-\xi}{c_f}}}{2c_f}\, u(\xi)\, d\xi, \qquad r \ge r_e\,. \end{cases} \tag{13}$$

Eq. $(13)_1$ (Eq. $(13)_2$) evaluated for $r = r_i$ ($r = r_e$) is equal to Eq. (10) evaluated for $r = r_i$ ($r = r_e$). The same results can be achieved by computing the first derivative of Eq. $(13)_1$ (Eq. $(13)_2$) evaluated for $r = r_i$ ($r = r_e$) and comparing it with the derivative of Eq. (10) evaluated at $r = r_i$ ($r = r_e$). Therefore, no discontinuities arise. However, evaluation of the reaction field outside the structural domain has no physical meaning and can be thus disregarded without affecting in any way the structural problem solved for $r \in [r_i, r_e]$.

5. Nonlocal free vibration problem

The differential condition of d'Alembert dynamic equilibrium for nanoplates on nanofoundations writes as

$$\partial_r^2\big(M_r(r,t)\cdot r\big) - \partial_r M_\theta(r,t) = r\left(q(r,t) - f(r,t) - \rho\, h\, \partial_t^2 u(r,t)\right), \tag{14}$$

being ρ the mass density, $\rho\, h$ the mass per unit area and t the time variable. The partial differential equation governing equilibrium of a freely vibrating nanoplate can be derived by setting $q = 0$ in Eq. (14). Explicit expression of the circumferential bending interaction field M_θ is obtained by exploiting Eq. (5), i.e.

$$M_\theta(r,t) = D\,\left(\chi_\theta(r,t) + \nu\, \chi_r(r,t)\right)\,, \tag{15}$$

while the radial bending interaction field M_r is derived from the following differential equation

$$\chi_r(r,t) - l_c^2 \partial_r^2 \chi_r(r,t) = \frac{M_r(r,t) - \nu\, M_\theta(r,t)}{D\,(1-\nu^2)}\,. \tag{16}$$

The radial and circumferential curvature fields in Eqs. (15)–(16) are explicitly expressed by $\chi_r(r,t) = \partial_r^2 u(r,t)$ and $\chi_\theta(r,t) = \frac{1}{r}\partial_r u(r,t)$. The foundation reaction field f is then obtained from Eq. (14) and substituted into Eq. (11) to derive the governing partial differential equation, which can be written by separating the time and space variables $u(r,t) = \phi(t)\, w(r)$, i.e.

$$\frac{\ddot{\phi}(t)}{\phi(t)} = \frac{\mathcal{F}(w(r),\ldots,w^{(8)}(r))}{h\,\rho\left(w(r) - c_f^2\, w''(r)\right)} = -\omega^2, \tag{17}$$

with

$$\begin{aligned}
\mathcal{F}(w(r),\ldots,w^{(8)}(r)) :=\; & \frac{D(-12c_f^2 + r^2)}{r^4}\, w''(r) + \frac{D(9c_f^2 - 2r^2)}{r^3}\, w^{(3)}(r) - \frac{D(5c_f^2 r + r^3)}{r^3}\, w^{(4)}(r) - \frac{4c_f^2\, D\, l_c^2}{r^3}\, w^{(5)}(r) \\
& + \frac{2c_f^2 D}{r}\, w^{(5)}(r) + \frac{2D\, l_c^2}{r}\, w^{(5)}(r) + \frac{4c_f^2\, D\, l_c^2 \nu^2}{r^3}\, w^{(5)}(r) - \frac{2D\, l_c^2\, \nu^2}{r}\, w^{(5)}(r) + c_f^2\, D\, w^{(6)}(r) + D\, l_c^2\, w^{(6)}(r) + \frac{4c_f^2\, D\, l_c^2}{r^2}\, w^{(6)}(r) \\
& - D\, l_c^2\, \nu^2\, w^{(6)}(r) - \frac{4\, c_f^2\, D\, l_c^2\, \nu^2}{r^2}\, w^{(6)}(r) - \frac{2c_f^2\, D\, l_c^2}{r}\, w^{(7)}(r) + \frac{2c_f^2\, D\, l_c^2\, \nu^2}{r}\, w^{(7)}(r) - c_f^2\, D\, l_c^2\, w^{(8)}(r) + c_f^2\, D\, l_c^2\, \nu^2\, w^{(8)}(r) \\
& - k_W\, w(r) - \frac{D(-12c_f^2 + r^2)}{r^5}\, w'(r)\,.
\end{aligned} \tag{18}$$

Time differentiation is indicated by an overdot, whereas derivatives with respect to the spatial coordinate r are represented by primes. Higher-order derivatives are written as $w^{(n)}$, $n \in \{3, \ldots, 8\}$. Eq. (17) can be equivalently recast as the following system

$$\ddot{\phi}(t) + \omega^2 \phi(t) = 0\,, \tag{19a}$$

$$\mathcal{F}(w(r),\ldots,w^{(8)}(r)) + \omega^2 h\,\rho\left(w(r) - c_f^2\, w''(r)\right) = 0\,. \tag{19b}$$

While the solution to the harmonic motion Eq. (19a) is given by $\phi(t) = A\sin(\omega t) + B\cos(\omega t)$, with the constants A, B depending on the prescribed initial conditions and ω the natural angular frequency, Eq. (19b) requires further manipulation in order to investigate its solutions. To this aim, the non-dimensional radius $\tilde{r} = \frac{r}{r_e}$ is defined along with the following nondimensional parameters

$$\lambda_b = \frac{l_c}{r_e}\,, \quad \lambda_f = \frac{c_f}{r_e}\,, \quad \tilde{w} = \frac{w}{r_e}\,, \quad \Omega^2 = \frac{r_e^4}{D}\, h\rho\,\omega^2\,, \quad K_W = \frac{r_e^4}{D}\, k_W\,. \tag{20}$$

Rewriting Eq. (19b) in terms of the nondimensional variables defined above, and omitting the tildes for simplicity, yields

$$\begin{aligned}
& - K_W\, w(r) + \Omega^2 w(r) - \frac{w'(r)}{r^3} + \frac{12\lambda_f^2}{r^5}\, w'(r) + \frac{w''(r)}{r^2} - \frac{12\lambda_f^2}{r^4}\, w''(r) - \lambda_f^2 \Omega^2 w''(r) - \frac{2}{r}\, w^{(3)}(r) + \frac{9\lambda_f^2}{r^3}\, w^{(3)}(r) \\
& - w^{(4)}(r) - \frac{5\lambda_f^2}{r^2}\, w^{(4)}(r) + \frac{2\lambda_b^2}{r}\, w^{(5)}(r) + \frac{2\lambda_f^2}{r}\, w^{(5)}(r) - \frac{4\lambda_b^2\lambda_f^2}{r^3}\, w^{(5)}(r) - \frac{2\lambda_b^2\nu^2}{r}\, w^{(5)}(r) + \frac{4\lambda_b^2\lambda_f^2\nu^2}{r^3}\, w^{(5)}(r) \\
& + \lambda_b^2 w^{(6)}(r) + \lambda_f^2 w^{(6)}(r) + \frac{4\lambda_b^2\lambda_f^2}{r^2}\, w^{(6)}(r) - \lambda_b^2\nu^2 w^{(6)}(r) - \frac{4\lambda_b^2\lambda_f^2\nu^2}{r^2}\, w^{(6)}(r) - \frac{2\lambda_b^2\lambda_f^2}{r}\, w^{(7)}(r) + \frac{2\lambda_b^2\lambda_f^2\nu^2}{r}\, w^{(7)}(r) \\
& - \lambda_b^2\lambda_f^2 w^{(8)}(r) + \lambda_b^2\lambda_f^2\nu^2 w^{(8)}(r) = 0\,.
\end{aligned} \tag{21}$$

Eq. (21) is an 8th-order nonautonomous differential equation which does not admit a closed form expression for the generalized solution depending on suitable constants c_i, $i \in \{1,\ldots,8\}$. Therefore, the standard strategy adopted to determine the first natural frequencies – consisting in rewriting the boundary conditions in terms of the generalized solution and thus obtaining a homogeneous algebraic linear system of 8 equations for the unknown constants c_i and finding the values of Ω satisfying the characteristic nonlinear equation for which the determinant of the supporting matrix vanishes – is not applicable here. In order to compute the natural frequencies and obtain the corresponding normalized mode shapes, we adopt the computational package `Compound Matrix Method` provided by Pearce (2025) based on the Wolfram language. This method allows to determine the eigenvalues of boundary-value ordinary differential equations as follows. First, a transformation of the boundary-value problem into a set of first-order matrix equations is performed, then the Evans function is constructed and evaluated at a prescribed guess of the eigenvalue. The eigenvalues of the original boundary-value problem are then given by the zeros of the Evans function.

6. Case studies

In the following investigation, numerical values of the dimensional parameters are fixed as shown in Table 1, referring to a single-layered graphene sheet (see Farajpour et al. (2021)). According to the nondimensional parameter in Eq. (20), the nondimensional internal and external radii are 0.125 and 1, respectively, while the nondimensional stiffness parameter is set as $K_W = 10$.

Table 1
Parameters, descriptions, units and values set.

Parameter	Description	Unit	Value
r_i	Internal radius	nm	2.5
r_e	External radius	nm	20
h	Thickness	nm	0.34
ν	Poisson ratio	–	0.25
E	Euler-Young modulus	GPa	10^3
ρ	Mass density	kg m^{-3}	2250

Two case studies are investigated, namely a pinned-clamped and clamped nanoplate on nonlocal medium. In both cases, the nondimensional form of the four constitutive boundary conditions in Eqs. (9) and (12) is prescribed, namely

$$\left\{\begin{aligned}
&\lambda_b\, w^{(3)}(r_i) - w''(r_i) = 0\,,\\
&\lambda_b\, w^{(3)}(r_e) + w''(r_e) = 0\,,\\
&\frac{w(r_i)}{\lambda_f} - w'(r_i) + \frac{1}{\Omega^2}\Bigg(-\frac{3\,w'(r_i)}{r_i^4} - \frac{w'(r_i)}{r_i^3\lambda_f} + \frac{3\,w''(r_i)}{r_i^3} + \frac{w''(r_i)}{r_i^2\lambda_f} - \frac{3\,w^{(3)}(r_i)}{r_i^2} - \frac{2\,w^{(3)}(r_i)}{r_i\lambda_f}\\
&+\frac{2\,w^{(4)}(r_i)}{r_i} - \frac{w^{(4)}(r_i)}{\lambda_f} + w^{(5)}(r_i) + \frac{2\lambda_b^2\,w^{(5)}(r_i)}{r_i^2} + \frac{2\lambda_b^2\,w^{(5)}(r_i)}{r_i\lambda_f} - \frac{2\lambda_b^2\nu^2\,w^{(5)}(r_i)}{r_i^2} - \frac{2\lambda_b^2\nu^2\,w^{(5)}(r_i)}{r_i\lambda_f}\\
&-\frac{2\lambda_b^2\,w^{(6)}(r_i)}{r_i} + \frac{\lambda_b^2\,w^{(6)}(r_i)}{\lambda_f} + \frac{2\lambda_b^2\nu^2\,w^{(6)}(r_i)}{r_i} - \frac{\lambda_b^2\nu^2\,w^{(6)}(r_i)}{\lambda_f} - \lambda_b^2\,w^{(7)}(r_i) + \lambda_b^2\nu^2\,w^{(7)}(r_i)\Bigg) = 0\,,\\
&-\frac{w(r_e)}{\lambda_f} - w'(r_e) + \frac{1}{\Omega^2}\Bigg(-\frac{3\,w'(r_e)}{r_e^4} + \frac{w'(r_e)}{r_e^3\lambda_f} + \frac{3\,w''(r_e)}{r_e^3} - \frac{w''(r_e)}{r_e^2\lambda_f} - \frac{3\,w^{(3)}(r_e)}{r_e^2} + \frac{2\,w^{(3)}(r_e)}{r_e\lambda_f}\\
&+\frac{2\,w^{(4)}(r_e)}{r_e} + \frac{w^{(4)}(r_e)}{\lambda_f} + w^{(5)}(r_e) + \frac{2\lambda_b^2\,w^{(5)}(r_e)}{r_e^2} - \frac{2\lambda_b^2\,w^{(5)}(r_e)}{r_e\lambda_f} - \frac{2\lambda_b^2\nu^2\,w^{(5)}(r_e)}{r_e^2} + \frac{2\lambda_b^2\nu^2\,w^{(5)}(r_e)}{r_e\lambda_f}\\
&-\frac{2\lambda_b^2\,w^{(6)}(r_e)}{r_e} - \frac{\lambda_b^2\,w^{(6)}(r_e)}{\lambda_f} + \frac{2\lambda_b^2\nu^2\,w^{(6)}(r_e)}{r_e} + \frac{\lambda_b^2\nu^2\,w^{(6)}(r_e)}{\lambda_f} - \lambda_b^2\,w^{(7)}(r_e) + \lambda_b^2\nu^2\,w^{(7)}(r_e)\Bigg) = 0\,.
\end{aligned}\right. \tag{22}$$

In addition to the constitutive boundary conditions in Eq. (22), the following essential and natural boundary conditions are prescribed for a plate simply supported at $r = r_i$ and fixed at $r = r_e$

$$\left\{\begin{aligned}
&w(r_i) = 0\,,\\
&\nu\frac{w'(r_i)}{r_i} + w''(r_i) - \lambda_b^2\,w^{(4)}(r_i) + \lambda_b^2\nu^2\,w^{(4)}(r_i) = 0\,,\\
&w(r_e) = 0\,,\\
&w'(r_e) = 0\,.
\end{aligned}\right. \tag{23}$$

For a fixed plate the following four essential boundary conditions are prescribed, in addition to the constitutive boundary conditions in Eq. (22), i.e.

$$\left\{\begin{aligned}
&w(r_i) = 0\,,\\
&w'(r_i) = 0\,,\\
&w(r_e) = 0\,,\\
&w'(r_e) = 0\,.
\end{aligned}\right. \tag{24}$$

The outcomes obtained using the `Compound Matrix Method` described in Section 5 are shown below, assuming that $\lambda_b = \lambda_f = \lambda$. The first four nondimensional natural frequencies $\bar{\Omega}_i := \dfrac{\Omega_i}{\Omega_{l_i}}$, $i \in \{1, \ldots, 4\}$ are provided for both the pinned-clamped and clamped plate, as summarized in Tables 2 and 3, respectively, where Ω_{l_i} denotes the corresponding local natural frequency. It is worth noting that for any fixed value of λ the natural frequencies in the fixed plate case are higher than the ones obtained for the pinned-fixed case.

The normalized eigenfunctions Ψ_i, $i \in \{1, \ldots, 4\}$ are plotted as functions of the nondimensional radial variable r in Figs. 1–2 for the pinned-clamped and clamped plate case, respectively. Analogously, the normalized eigenfunctions are displayed as functions of the two variables $x = r\cos\theta$, $y = r\sin\theta$ with $\theta \in [0, \pi]$ in Figs. 3–4 for the pinned-clamped and clamped plate, respectively.

7. Closing remarks

A consistent nonlocal methodology has been developed to investigate the free vibration behavior of axisymmetric nanoplates resting on elastic nanofoundations. The size dependent mechanical response of the nanoplate has been described exploiting the stress-driven integral elasticity theory, while the interaction with the supporting medium has been modeled through a displacement-driven

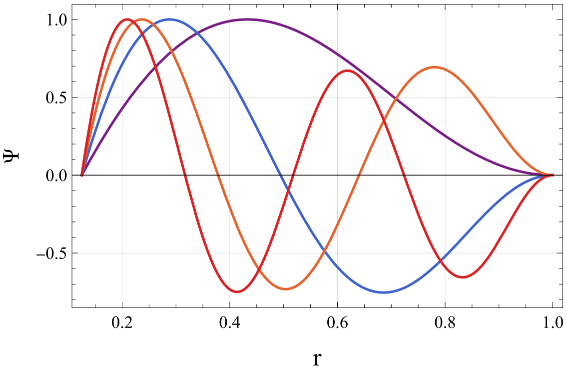


Fig. 1. Plate with pinned and clamped edges: first four eigenfunctions for $\lambda = 0.5$.

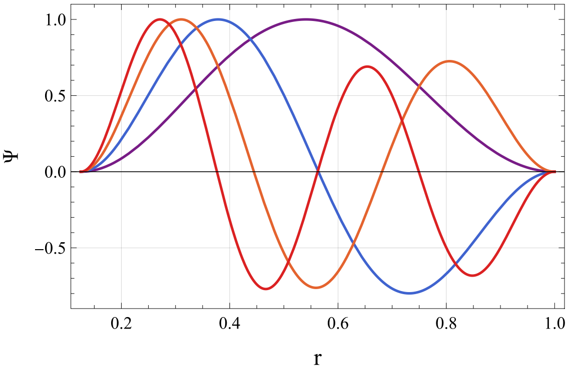


Fig. 2. Plate with clamped edges: first four eigenfunctions for $\lambda = 0.5$.

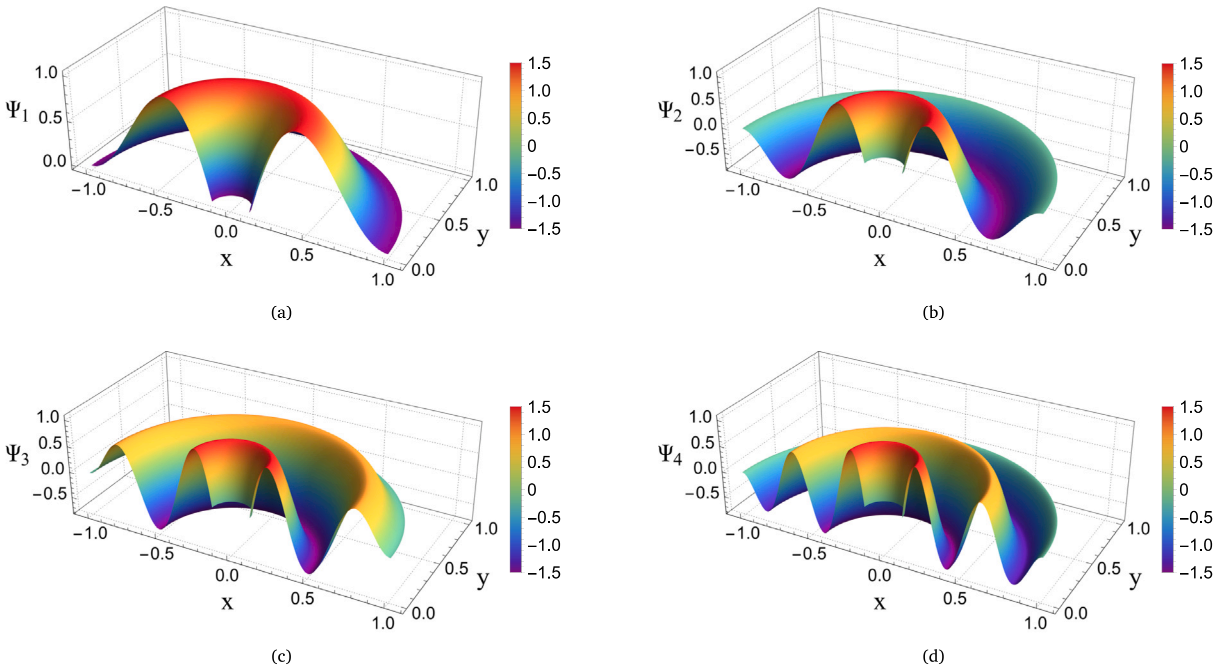


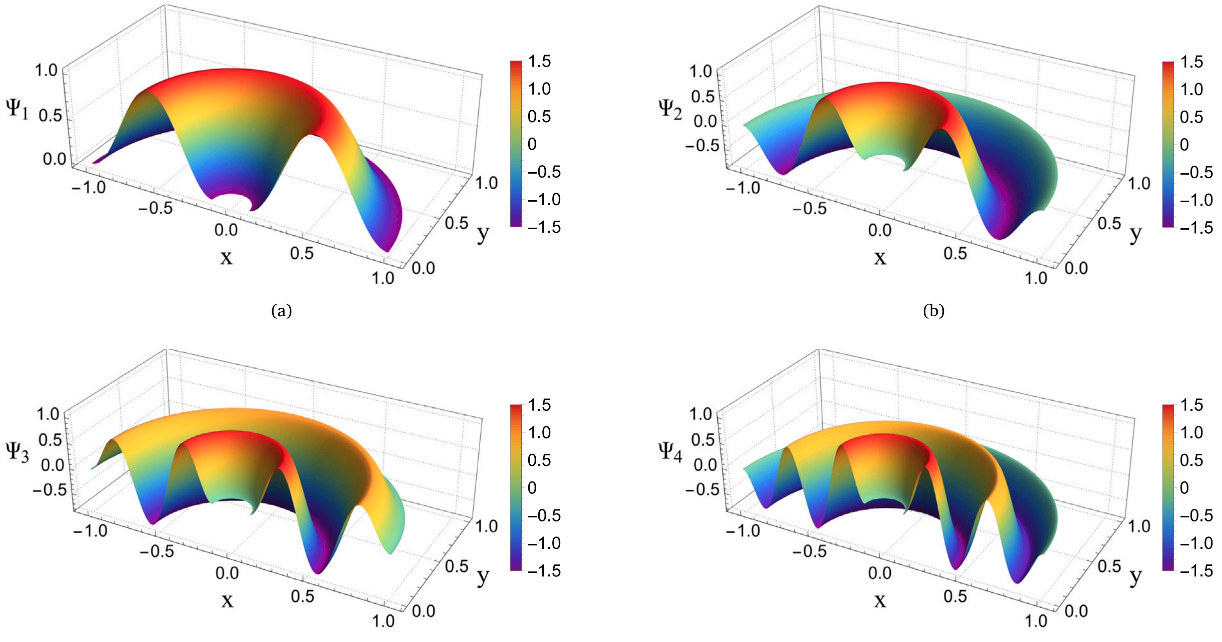
Fig. 3. Plate with pinned and clamped edges: first four eigenfunctions as functions of $x = r\cos\theta$, $y = r\sin\theta$ with $\theta \in [0, \pi]$ for $\lambda = 0.5$.

Table 2
Plate with pinned and clamped edges: nondimensional natural frequencies $\bar{\Omega}_i := \dfrac{\Omega_i}{\Omega_{l_i}}$, $i \in \{1, \ldots, 4\}$.

λ	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$
0.1	1.171	1.339	1.563	1.824
0.2	1.432	1.911	2.485	3.105
0.3	1.731	2.549	3.477	4.453
0.4	2.047	3.209	4.492	5.820
0.5	2.373	3.880	5.515	7.196

Table 3
Plate with clamped edges: nondimensional natural frequencies $\bar{\Omega}_i := \dfrac{\Omega_i}{\Omega_{l_i}}$, $i \in \{1, \ldots, 4\}$.

λ	$\bar{\Omega}_1$	$\bar{\Omega}_2$	$\bar{\Omega}_3$	$\bar{\Omega}_4$
0.1	1.342	1.545	1.792	2.071
0.2	1.874	2.429	3.037	3.680
0.3	2.477	3.397	4.362	5.362
0.4	3.108	4.392	5.712	7.066
0.5	3.753	5.400	7.072	8.778



(a) (b)

(c) (d)

Fig. 4. Plate with clamped edges: first four eigenfunctions as functions of $x = r\cos\theta$, $y = r\sin\theta$ with $\theta \in [0, \pi]$ for $\lambda = 0.5$.

nonlocal elastic approach. This combined methodology allows for modeling long-range interactions in both the structural element and the foundation, overcoming paradoxes and intrinsic difficulties associated with traditional formulations.

By adopting Helmholtz averaging kernels, the integral constitutive laws governing the plate and the foundation size dependent behaviors have been equivalently transformed into differential constitutive problems. This strategy leads to a high-order boundary-value problem that accurately captures size-dependent effects. The Compound Matrix Method has been applied to numerically compute natural frequencies and corresponding mode shapes. The numerical investigations performed for different case studies have shown that both the plate and foundation characteristic length parameters significantly affect the vibrational response. The results confirm that the proposed formulation provides a mathematically consistent and physically meaningful tool for the dynamic analysis of nanoplates interacting with elastic media and can be effectively used in the modeling and design of advanced nanoscale structural systems.

CRediT authorship contribution statement

Marzia Sara Vaccaro: Writing – original draft, Software, Methodology, Investigation, Formal analysis, Conceptualization. **Annalisa Iuorio:** Writing – original draft, Software, Methodology, Investigation. **Andrea Caporale:** Writing – original draft, Methodology, Investigation, Conceptualization. **Raffaele Barretta:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Raimondo Luciano:** Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

References

Alnujaie, A. H., Daikh, A. A., Ghazwani, M. H. H., Assie, A. E., & Eltaher, M. A. (2024). Size-dependent free vibration of coated functionally graded graphene reinforced nanoplates rested on viscoelastic medium. *Advances in Nano Research*, *17*(2), 181–195.

Bai, M., Zhai, Y., Liu, F., Wang, X., et al. (2019). Stretchable graphene thin film enabled yarn sensors with tunable piezoresistivity for human motion monitoring. *Scientific Reports*, *9*, 18644.

Barchiesi, E., dell'Isola, F., & Hild, F. (2021). On the validation of homogenized modeling for bipantographic metamaterials via digital image correlation. *International Journal of Solids and Structures*, *208*, 49–62. http://dx.doi.org/10.1016/j.ijsolstr.2020.11.003.

Barretta, R., Čanađija, M., Luciano, R., & Marotti de Sciarra, F. (2022). On the mechanics of nanobeams on nano-foundations. *International Journal of Engineering Science*, *180*, Article 103747.

Barretta, R., Faghidian, S. A., & Marotti de Sciarra, F. (2019). Stress-driven nonlocal integral elasticity for axisymmetric nano-plates. *International Journal of Engineering Science*, *136*, 38–52.

Barretta, R., Marotti de Sciarra, F., & Vaccaro, M. S. (2023). Nonlocal elasticity for nanostructures: A review of recent achievements. *Encyclopedia*, *3*, 279–310. http://dx.doi.org/10.3390/encyclopedia3010018.

Batool, F., Muhammad, S., Muazzam, R., Waqas, M., Ullah, Z., Roy, S., Zhang, Y., Wang, K., & Guo, B. (2025). Advancements in graphene-based composites: A review of the emerging applications in healthcare. *Smart Materials in Medicine*, *6*(1), 120–138.

Behnam-Rasouli, M.-S., Challamel, N., Karamodin, A., & Sani, A. A. (2024). Application of the Green's function method for static analysis of nonlocal stress-driven and strain gradient elastic nanobeams. *International Journal of Solids and Structures*, *295*, Article 112794.

Benvenuti, E., & Simone, A. (2013). One-dimensional nonlocal and gradient elasticity: Closed-form solution and size effect. *Mechanics Research Communications*, *48*, 46–51.

Caporale, A., Luciano, R., Scorza, D., & Vantadori, S. (2023). Local–nonlocal stress-driven model for multi-cracked nanobeams. *International Journal of Solids and Structures*, *273*, Article 112230.

Capurso, M. (1967). A generalization of wieghardt soil for two - dimensional foundation structures. *Meccanica*, *2*(1), 49–54.

Chen, G., Lin, T., Lodewijks, G., & Ji, A. (2023). Design of an active flexible spine for wall climbing robot using pneumatic soft actuators. *Journal of Bionic Engineering*, *20*(2), 530–542.

Chu, T., Nguyen, V. L., Le, T. T., Tu, T., & Dang, X. (2026). Nonlocal vibration response of FG Saturated Porous Nanoplate Resting on Elastic Foundation Using Quasi-3D HSDT theory. *Advances in Applied Mathematics and Mechanics*, *18*(2), 561–600.

Cianci, R., Bruzzone, A. G., Sburlati, R., & Jafarinezhad, M. (2024). Stress-driven simulation model in polar coordinates: The analytical case for circular plates. In *Proceedings of the 23rd international conference on modeling & applied simulation* (p. 020). http://dx.doi.org/10.46354/i3m.2024.mas.020.

Cianci, R., Jafarinezhad, M., & Sburlati, R. (2025). A Gauss kernel non-local stress-driven plate theory. *Composite Structures*, *351*.

Daikh, A. A., Belarbi, M. O., Drai, A., Houari, M. S. A., Avcar, M., Tounsi, A., & Eltaher, M. A. (2023). Static analysis of functionally graded plate structures resting on variable elastic foundation under various boundary conditions. *Acta Mechanica*, *234*(2), 775–806.

Das, B., Ussorio, D., Vaccaro, M. S., Barretta, R., & Luciano, R. (2025). Dynamics of FG nanobeams on nonlocal medium. *Composite Structures*, *366*.

dell'Isola, F., Giorgio, I., Pawlikowski, M., & Rizzi, N. L. (2016). Large deformations of planar extensible beams and pantographic lattices: heuristic homogenization, experimental and numerical examples of equilibrium. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *472*(2185), Article 20150790. http://dx.doi.org/10.1098/rspa.2015.0790.

Eringen, A. C. (1972). Linear theory of nonlocal elasticity and dispersion of plane waves. *International Journal of Engineering Science*, *10*, 425–435.

Eringen, A. C. (1983). On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves. *Journal of Applied Physics*, *54*(9), 4703–4710.

Eroğlu, M., Esen, İ., & Koç, M. (2025). The effect of an auxetic core layer and symmetric FGM face layers on the 3D wave propagation response of sandwich nanoplates. *Archive of Applied Mechanics*, *95*(1), 69.

Farajpour, A., Żur, K. K., Kim, J., & Reddy, J. (2021). Nonlinear frequency behaviour of magneto-electromechanical mass nanosensors using vibrating MEE nanoplates with multiple nanoparticles. *Composite Structures*, *260*, Article 113458.

Fernández-Sáez, J., Zaera, R., Loya, J., & Reddy, J. (2016). Bending of Euler–Bernoulli beams using Eringen's integral formulation: A paradox resolved. *International Journal of Engineering Science*, *99*, 107–116.

Hozhabrossadati, S. M., Challamel, N., Rezaiee-Pajand, M., & Sani, A. A. (2018). Application of Green's function method to bending of stress gradient nanobeams. *International Journal of Solids and Structures*, *143*, 209–217.

Indronil, D. (2025). Dynamics of nonlocal stress-driven Rayleigh Beam. *Forces in Mechanics*, *18*, Article 100299.

Izadi, R., Das, R., Fantuzzi, N., & Trovalusci, P. (2024). Fracture properties of green nano fibrous network with random and aligned fiber distribution: A hierarchical molecular dynamics and peridynamics approach. *International Journal of Engineering Science*, *204*.

Jafarinezhad, M., Sburlati, R., & Cianci, R. (2023a). Static and free vibration analysis of functionally graded annular plates using stress-driven nonlocal theory. *European Journal of Mechanics. A. Solids*, *99*, Article 104955.

Jafarinezhad, M., Sburlati, R., & Cianci, R. (2023b). Static and free vibration analysis of functionally graded annular plates using stress-driven nonlocal theory. *European Journal of Mechanics. A. Solids*, *99*.

Jafarinezhad, M., Sburlati, R., & Cianci, R. (2024). Nonlocal stress-driven model for functionally graded Mindlin annular plate: bending and vibration analysis. *Archive of Applied Mechanics*, *94*(5), 1313–1333.

Khodabakhshi, P., & Reddy, J. (2015). A unified integro-differential nonlocal model. *International Journal of Engineering Science*, *95*, 60–75, URL: https://www.sciencedirect.com/science/article/pii/S0020722515000919.

Kong, M., Yang, M., Li, R., et al. (2024). Graphene-based flexible wearable sensors: Mechanisms, challenges, and future directions. *International Journal of Advanced Manufacturing Technology*, *131*, 3205–3237.

Kröner, E. (1967). Elasticity theory of materials with long range cohesive forces. *International Journal of Solids and Structures*, *3*(5), 731–742.

Krumhansl, J. A. (1968). Some considerations of the relation between solid state physics and generalized continuum mechanics. In E. Kröner (Ed.), *Mechanics of generalized continua* (pp. 298–311). Berlin, Heidelberg: Springer Berlin Heidelberg.

Kunin, I. A. (1968). The theory of elastic media with microstructure and the theory of dislocations. In E. Kröner (Ed.), *Mechanics of generalized continua* (pp. 321–329). Berlin, Heidelberg: Springer Berlin Heidelberg.

van Langendonck, T. (1962). Beams on deformable foundation. *Memoires A.I.P.C.*, *22*, 113–128.

Li, Y. S., Li, A., Zhou, S., Zhang, R., Wei, G., Ren, F., Qiao, J., & Zhou, S. (2025). Analysis of mechanical properties of flexoelectric nano-plates based on nonlocal strain gradient theory. *Mechanics of Solids*, *60*(2), 1272–1288, Cited by: 0.

Lovisi, G. (2023). Application of the surface stress-driven nonlocal theory of elasticity for the study of the bending response of FG cracked nanobeams. *Composite Structures*, *324*, Article 117549.

Lovisi, G., Feo, L., Lambiase, A., & Penna, R. (2024). Application of surface stress-driven model for higher vibration modes of functionally graded nanobeams. *Nanomaterials*, *14*(4), 350.

Momeni-Khabisi, H., & Tahani, M. (2026). Size-dependent thermo-mechanical stability of flexomagnetic nano-plates with initial imperfections. *European Journal of Mechanics. A. Solids*, *117*, Cited by: 0.

Nguyen, H., Challamel, N., & Wang, C. (2025). Analytical solutions for free vibrations of rectangular cuboid elastic lattices and their continuous approximations. *Journal of Sound and Vibration*, *600*, Article 118836.

Özdemir, D. G., Evcimen Duygulu, N., Özarslan, A. C., & Ciftci, F. (2024). Fabrication and characterization of Graphene oxide/Fucoidan/Chitosan reinforced Poly(vinyl alcohol) nanocomposites. *Journal of Molecular Structure*, *1301*, Article 137330.

Pasternak, P. (1954). *On a new method of analysis of an elastic foundation by means of two foundation constants*.

Pearce, S. (2025). SPPearce/CompoundMatrixMethod: v0.92.1. http://dx.doi.org/10.5281/zenodo.15849658.

Peddieson, J., Buchanan, G. R., & McNitt, R. P. (2003). Application of nonlocal continuum models to nanotechnology. *International Journal of Engineering Science*, *41*(3–5), 305–312.

Penna, R., & Lovisi, G. (2025). Surface and nonlocal effects in piezoelectric nanobeams. *European Journal of Mechanics. A. Solids*, *113*, Article 105715.

Placidi, L., dell'Isola, F., Kandalaft, A., Luciano, R., Majorana, C., & Misra, A. (2024). A granular micromechanic-based model for ultra high performance fiber-reinforced concrete (UHP FRC). *International Journal of Solids and Structures*, *297*, Article 112844.

Placidi, L., Greco, L., Bucci, S., Turco, E., & Rizzi, N. (2016). A second gradient formulation for a 2D fabric sheet with inextensible fibres. *Zeitschrift für Angewandte Mathematik und Physik*, *67*, http://dx.doi.org/10.1007/s00033-016-0701-8.

Placidi, L., Sherbiny, M. G. E., & Baragatti, P. (2022). Experimental investigation for the existence of frequency band gap in a microstructure model. *Mathematics and Mechanics of Complex Systems*, *9*(4), 413–421.

Reissner, E. (1958). A note on deflections of plates on a viscoelastic foundation. *Journal of Applied Mechanics*, *25*(1), 144–145.

Rezaei, A., Izadi, R., & Fantuzzi, N. (2025). A non-classical computational method for modelling functionally graded porous planar media using micropolar theory. *Computers and Structures*, *306*.

Rezaiee-Pajand, M., & Rajabzadeh-Safaei, N. (2022). Stress-driven nonlinear behavior of curved nanobeams. *International Journal of Engineering Science*, *178*, Article 103724.

Rogula, D. (1965). Influence of spatial acoustic dispersion on dynamical properties of dislocations. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, *13*, 337–385.

Rogula, D. (1982). Introduction to nonlocal theory of material media. In D. Rogula (Ed.), *Nonlocal theory of material media* (pp. 123–222). Vienna: Springer Vienna.

Romano, G., & Barretta, R. (2017). Nonlocal elasticity in nanobeams: the stress-driven integral model. *International Journal of Engineering Science*, *115*, 14–27.

Romano, G., Barretta, R., & Diaco, M. (2017). On nonlocal integral models for elastic nano-beams. *International Journal of Mechanical Sciences*, *131–132*, 490–499.

Romano, G., Barretta, R., Diaco, M., & de Sciarra, F. M. (2017). Constitutive boundary conditions and paradoxes in nonlocal elastic nanobeams. *International Journal of Mechanical Sciences*, *121*, 151–156.

Simões, F. M. F., & Pinto da Costa, A. (2026). Dynamics of a beam on a displacement-driven nonlocal foundation under a moving load. *International Journal of Solids and Structures*, *330*, Article 113877.

Sollazzo, A. (1966). Equilibrio della trave su suolo di Wieghardt. *Tecnica Italiana*, *31*(4), 187–206.

Vaccaro, M. S., Pinnola, F. P., Marotti de Sciarra, F., & Barretta, R. (2021). Elastostatics of Bernoulli–Euler beams resting on displacement-driven nonlocal foundation. *Nanomaterials*, *11*, 573.

Vaccaro, M. S., & Sedighi, H. M. (2023). Two-phase elastic axisymmetric nanoplates. *Engineering with Computers*, *39*, 827–834.

Wang, S., Ding, W., Li, Z., Xu, B., Zhai, C., Kang, W., Yang, W., & Li, Y. (2023). A size-dependent quasi-3D model for bending and buckling of porous functionally graded curved nanobeam. *International Journal of Engineering Science*, *193*, Article 103962.

Wang, X., Yu, J., Zhang, B., Elmaimouni, L., & Li, L. (2025). Lamb waves in piezoelectric quasicrystal multi-layered nano-plates with imperfect interfaces. *Composite Structures*, *370*, Cited by: 1.

Wieghardt, K. (1922). Über den Balken auf nachgiebiger Unterlage. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, *2*(3), 165–184.

Winkler, E. (1867). *Die lehre von der elastizitat und festigkeit*. Prague.

Yıldırım, E., & Esen, I. (2025). Thermal vibration buckling of smart magneto electro elastic sandwich nano plates with biocompatible FGM core layer. *ZAMM Zeitschrift für Angewandte Mathematik und Mechanik*, *105*(12).

Zhang, P., Schiavone, P., & Qing, H. (2022). Stress-driven local/nonlocal mixture model for buckling and free vibration of FG sandwich Timoshenko beams resting on a nonlocal elastic foundation. *Composite Structures*, *289*, Article 115473.