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On the reduced Euler characteristic of independence complexes of circulant graphs

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ABSTRACT

Let G be the circulant graph $C_n(S)$ with $S \subseteq \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. We study the reduced Euler characteristic $\tilde{\chi}$ of the independence complex $\Delta(G)$ for $n = p^k$ with p prime and for $n = 2p^k$ with p odd prime, proving that in both cases $\tilde{\chi}$ does not vanish. We also give an example of circulant graph whose independence complex has $\tilde{\chi}$ which equals 0, giving a negative answer to R. Hoshino.

0. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset C of $V(G)$ is a *clique* of G if any two different vertices of C are adjacent in G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent in G. The *complement graph* of G, $\bar{G}$, is the graph with vertex set $V(G)$ and edge set $E(\bar{G}) = \{\{u, v\} \text{ with } u, v \in V(G) \mid \{u, v\} \notin E(G)\}$. In particular, a set is independent in G if and only if it is a clique in the complement graph $\bar{G}$.

We also recall that a circulant graph is defined as follows. Let $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $G := C_n(S)$ is a simple graph with $V(G) = \mathbb{Z}_n = \{0, \dots, n-1\}$ and $E(G) := \{\{i, j\} \mid |j-i|_n \in S\}$ where $|k|_n = \min\{|k|, n-|k|\}$.

Recently many authors have studied some combinatorial and algebraic properties of circulant graphs (see [7,3,2,12,5,10]). In particular, in [7,3,2,5], a formula for the f-vector of the independence complex was shown for some nice classes of circulants, e.g. the dth power cycle, $S = \{1, 2, \dots, d\}$, and its complement. Moreover, Hoshino in [7, p. 247] focused on the Euler characteristic, an invariant that is associated to any simplicial complex (see [4]). In particular, he conjectured, by our notation, that any independence complex associated to a non-empty circulant graph has reduced Euler characteristic always different from 0.

We show that for particular n, a circulant graph $C_n(S)$ will support the conjecture, independent of the choice of S. To this aim, we exploit that each entry of the f-vector is a multiple of a divisor of n (see Lemma 2.1).

In Section 2 we prove that the conjecture holds for $n = p^k$ for any prime p, and for $n = 2p^k$ for any odd prime p. Moreover we disprove the conjecture by providing a counterexample (see Example 2.10).

As an application of our results, we focus our attention on two algebraic objects related to the independence complex of circulant graphs. We first consider the *independence polynomial* (see [7,2]), that is

$$I(G, x) = \sum_{i=0}^{n} f_{i-1} x^i, \tag{0.1}$$

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where f_{i-1} are the entries of the f-vector of the independence complex of a graph G. In particular, -1 is a root of the independence polynomial if and only if the Euler characteristic of the independence complex vanishes. This happens in Example 2.10 and does not happen for all the cases studied in Theorems 2.3, 2.9.

The second application arises from commutative algebra (see e.g. [4,9,14,11]). Let $R = K[x_0, \dots, x_{n-1}]$ be the polynomial ring and $I(G)$ the edge ideal related to the graph G (see [13]), that is

$$I(G) = (x_i x_j : \{i, j\} \in E(G)). \tag{0.2}$$

In this case the non-vanishing of the reduced Euler characteristic gives us information about the regularity index of $R/I(G)$, namely the smallest integer such that the Hilbert function on $R/I(G)$ becomes a polynomial function, the so-called Hilbert polynomial (see Section 1, Remark 1.2). Also in this case Theorems 2.3, 2.9 and Example 2.10 are relevant.

1. Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article.

Set $V = \{x_1, \dots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that: 1) $\{x_i\} \in \Delta$ for all $x_i \in V$; 2) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.

The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of Δ is the maximum of the dimensions of all facets. Let $d - 1$ be the dimension of Δ and let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1} = 1$. Then the f-vector of Δ is the $(d+1)$-tuple $f(\Delta) = (f_{-1}, f_0, \dots, f_{d-1})$. The h-vector of Δ is $h(\Delta) = (h_0, h_1, \dots, h_d)$ with

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}. \tag{1.1}$$

The sum

$$\widetilde{\chi}(\Delta) = \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}$$

is called the *reduced Euler characteristic* of Δ and $h_d = (-1)^{d-1}\widetilde{\chi}(\Delta)$.

Given any simplicial complex Δ on V, we can associate a monomial ideal I_Δ in the polynomial ring R as follows:

$$I_\Delta = (\{x_{j_1} x_{j_2} \cdots x_{j_r} : \{x_{j_1}, x_{j_2}, \dots, x_{j_r}\} \notin \Delta\}).$$

R/I_Δ is called *Stanley–Reisner ring* and its Krull dimension is d. If G is a graph, the *independence complex* of G is

$$\Delta(G) = \{A \subset V(G) : A \text{ is an independent set of } G\}.$$

The independence polynomial is associated to $\Delta(G)$ and by Eq. (0.1) it follows

$$\widetilde{\chi}(\Delta(G)) = -I(G, -1). \tag{1.2}$$

We also remark that from the definition of Stanley–Reisner ring and by Eq. (0.2), it follows $R/I_{\Delta(G)} = R/I(G)$.

The *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G.

Remark 1.1. Let $G = C_n(S)$ be a circulant graph on $S \subseteq T := \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. We observe that the complement graph of G, namely $\bar{G}$, is a circulant graph on $\bar{S} := T \setminus S$. Moreover the clique complex of $\bar{G}$ is the independence complex of G, $\Delta(G)$.

We also recall some basic facts about the regularity index (see [14, Chapter 5]). Let R be a standard graded ring and I be a homogeneous ideal. The *Hilbert function* $\mathrm{H}_{R/I} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathrm{H}_{R/I}(k) := \dim_K (R/I)_k$$

where $(R/I)_k$ is the k-degree component of the gradation of R/I (see [13, Section 2.2]), while the Hilbert–Poincaré series of R/I is

$$\mathrm{HP}_{R/I}(t) := \sum_{k \in \mathbb{N}} \mathrm{H}_{R/I}(k) t^k.$$

By the Hilbert–Serre theorem, the Hilbert–Poincaré series of R/I is a rational function, in particular

$$\mathrm{HP}_{R/I}(t) = \frac{h(t)}{(1-t)^n}$$

for some $h(t) \in \mathbb{Z}[t]$. There exists a unique polynomial $P_{R/I}$ such that $\mathrm{H}_{R/I}(k) = P_{R/I}(k)$ for all $k \gg 0$. The minimum integer $k_0 \in \mathbb{N}$ such that $\mathrm{H}_{R/I}(k) = P_{R/I}(k)$ for all $k \geq k_0$ is called *regularity index* and we denote it by $\mathrm{ri}(R/I)$.

We end this section with the following

Remark 1.2. Let R/I_Δ be a Stanley–Reisner ring. Then

$$\mathrm{ri}(R/I_\Delta) = \begin{cases} 0 & \text{if } h_d = 0 \\ 1 & \text{if } h_d \neq 0. \end{cases}$$

Related to the regularity index is the a-invariant (see Chapter 5 of [14]), namely the degree of $\mathrm{HP}_{R/I}(t)$ as a rational function, that gives further information about other algebraic invariants.

2. Reduced Euler characteristic of the independence complex of some circulants

The goal of this section is to study the reduced Euler characteristic, $\widetilde{\chi}$, of the independence complex $\Delta(G)$ of circulant graphs by proving bounds on the maximum clique number $\omega(\bar{G})$. In [10] the author proves that $\widetilde{\chi}(\Delta(G)) \neq 0$ when n is a prime number. We generalize the result for $n = p^k$ for any prime p, and $n = 2p^k$ for any odd prime p. For the sake of completeness, we give the following lemma that has been stated in [10, Lemma 1].

Lemma 2.1. *Let G be a circulant graph on n vertices with maximum independent set of cardinality d. Let f_{i-1} be the number of independent sets of cardinality i, and $f_{i-1,0}$ the number of them containing the vertex 0, then the following property holds*

$$if_{i-1} = nf_{i-1,0} \text{ with } 0 \leq i \leq d.$$

Proof. Let us call $\mathcal{F}_{i-1} \subset \Delta$ the set of faces of dimension $i-1$, that is

$$\mathcal{F}_{i-1} = \{F_1, \dots, F_{f_{i-1}}\}.$$

Let $f_{i-1,j}$ be number of faces in $\mathcal{F}_{i-1}$ containing a given vertex $j = 0, \dots, n-1$. Since G is circulant $f_{i-1,j} = f_{i-1,0}$ for all $j \in \{0, \dots, n-1\}$. Let $A \in \mathbb{F}_2^{f_{i-1}\times n}$, $A = (a_{jk})$, be the *incidence matrix*, whose

$$a_{jk} = \begin{cases} 1 & \text{if the vertex } k-1 \in F_j \\ 0 & \text{otherwise.} \end{cases}$$

We observe that each row has exactly i 1-entries. Hence summing the entries of the matrix we have if_{i-1}. Moreover each column has exactly $f_{i-1,j}$ non-zero entries. That is $if_{i-1} = nf_{i-1,0}$. □

A useful bound on the maximum clique number for non-complete circulant graphs is given by the following

Lemma 2.2. *Let $G = C_n(S)$ be a non-complete circulant graph. Then*

$$\omega(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Proof. Suppose that $\omega(G) > \lfloor \frac{n}{2} \rfloor$. So there exists a clique F of cardinality $\lfloor \frac{n}{2} \rfloor + 1$. Let $r \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The set $\{v + r \;:\; v \in F\}$ contains $\lfloor \frac{n}{2} \rfloor + 1$ vertices so at least one of them belongs to F. Hence there exist $v, w \in F$, such that $w = v + r$. Since F is a clique $\{v, w = v + r\} \in E(G)$, that is $r \in S$. The latter works for any r, then we conclude

$$S = \left\{1, 2, \dots, \left\lfloor \frac{n}{2} \right\rfloor\right\},$$

so G is complete, and this contradicts our assumption. □

Thanks to Lemma 2.2, we prove the following

Theorem 2.3. *Let p be a prime and let G be a non-empty circulant graph on $n = p^k$ vertices with $k > 0$. Then $\tilde{\chi}(\Delta(G)) \neq 0$.*

Proof. Given G circulant graph with maximum independent set of cardinality d, by Lemma 2.1 it follows

$$if_{i-1} = p^k f_{i-1,0} \text{ with } 0 \leq i \leq d.$$

Since the graph G is not empty, its complement graph $\bar{G}$ is not complete. Hence by Lemma 2.2, we have that a maximum clique in $\bar{G}$ has cardinality $d < \frac{p^k}{2}$, namely f_{i-1} is a non-zero multiple of p for $1 \leq i \leq d$. Therefore

$$\widetilde{\chi}(\Delta(G)) = \sum_{i=1}^{d} (-1)^{i-1} f_{i-1} - 1 = pr - 1$$

with $r \in \mathbb{Z}$. By the primality of p, $\widetilde{\chi}(\Delta(G))$ is always non-zero. □

Before stating the theorem on the case $n = 2p^k$, we prove some properties that will be helpful.

Lemma 2.4. *Let $n = 2q$ for an odd $q > 1$ and let $G = C_n(S)$ be a non-complete circulant graph. Then $\omega(G) < q$ if and only if $\{2, 4, \dots, q-1\} \not\subseteq S$.*

Proof. $(\Rightarrow)$. By contraposition assume $\{2, 4, \dots, q-1\} \subseteq S$. We observe that the set $\{0, 2, 4, \dots, 2q-2\}$ is a clique of cardinality q. It negates the hypothesis.

$(\Leftarrow)$. Let $r \in \{2, 4, \dots, q-1\}$ be such that $r \notin S$. Let C be the set of vertices of a clique of G. We claim $|C| < q$. We partition the set of vertices $V(G) = \{0, 1, 2, \dots, 2q-1\}$ in the two sets

$$V_1 = \{2k \mid k = 0, \dots, q-1\} \text{ and } V_2 = \{2k+1 \mid k = 0, \dots, q-1\}.$$

We observe that $|V_1| = |V_2| = q$.

Claim. *V_1 (respectively V_2) contains at most $\frac{q-1}{2}$ vertices such that for each pair $v, w \in V_1$ we have $|v-w|_n \neq r$.*

Proof of the Claim. By contraposition, assume that we take a subset V' of cardinality $\frac{q-1}{2}+1$ of V_1 with the desired property, say

$$V' = \{v_1, v_2, \dots, v_{\frac{q-1}{2}+1}\}$$

and since $V' \subset V_1$, these are all even vertices. Now we take the set $V'' = \{v + r : v \in V'\}$. Since r is even, $V'' \subset V_1$ and $|V'| = |V''| = \frac{q-1}{2}+1$. Since $|V'| + |V''| > q$, then $V' \cap V'' \neq \emptyset$, so there exist $v, w \in V'$ such that $w = v + r$; hence, the set V' has not the desired property. The claim follows.

Thus $|C \cap V_1| \leq \frac{q-1}{2}$ and $|C \cap V_2| \leq \frac{q-1}{2}$, so that $|C| < q$. □

We give a generalization of Lemma 2.4 in the following

Lemma 2.5. *Let $n = rq$ for an odd $q > 1$. Let $G = C_n(S)$ be a non-complete circulant graph. Then:*

(1) *If $\left\{r, 2r, \dots, \frac{q-1}{2}r\right\} \not\subseteq S$, then $\omega(G) \leq \frac{n-r}{2}$.*
(2) *If $\omega(G) < q$, then $\left\{r, 2r, \dots, \frac{q-1}{2}r\right\} \not\subseteq S$.*

Proof. (1) The proof follows the steps of $(\Leftarrow)$ of Lemma 2.4. We assume $jr \notin S$ for some j, $1 \leq j \leq \frac{q-1}{2}$. In this case we consider the partitions V_i of $V(G)$

$$V_i = \{rk + i \mid k = 0, \dots, q-1\}$$

with $i = 0, \dots, r-1$. Let C be the set of vertices of a clique of G. By using similar arguments to the Claim inside the proof of Lemma 2.4, we can choose at most $\frac{q-1}{2}$ vertices within each V_i such that for each pair $v, w \in V_i$ we have $|v-w|_n \neq jr$. Hence for any $i \in \{0, \dots, r-1\}$, it follows that $|C \cap V_i| \leq \frac{q-1}{2}$. Since $r(\frac{q-1}{2}) = \frac{n-r}{2}$, at the end we get $|C| \leq \frac{n-r}{2}$.

(2) The same proof of Lemma 2.4 $(\Rightarrow)$ holds. □

Remark 2.6. We highlight that by plugging $r = 2$ in (1) and (2) of Lemma 2.5, we obtain the two implications of Lemma 2.4. It is the unique case of $n = rq$ such that $\frac{rq-r}{2}$, the bound in (1), is equal to $q-1$, the bound in (2).

For the sake of simplicity, in Proposition 2.7 and Example 2.8 we focus our attention on the clique complex of the graph.

Proposition 2.7. *Let $n = 2p^k$ for an odd prime p, with $k > 0$, and let $G = C_n(S)$ be a circulant graph. If f_{p^k-1}, the number of cliques of cardinality p^k, is non-zero then*

$$f_{p^k-1} \equiv 2 \bmod p.$$

In particular, if one of the following condition holds

(a) *$1 \notin S$,*
(b) *$1 \in S$ and there exists $t \in \{1, \dots, p^k\}$ with $\gcd(t, 2p) = 1$ such that $t \notin S$,*

then $f_{p^k-1} = 2$.

Proof. First suppose that the graph is complete. Since $f_{p^k-1} = \binom{2p^k}{p^k}$ and by Lucas' Theorem [8], we obtain

$$f_{p^k-1} = \binom{2p^k}{p^k} \equiv 2 \bmod p.$$

Now suppose that G is not complete. By Lemmas 2.2 and 2.4, since $f_{p^k-1} \neq 0$, that is $\omega(G) = p^k$, we have that $\{2, 4, \dots, p^k-1\} \subseteq S$. So $f_{p^k-1} \geq 2$. In fact the graph contains at least the two maximal cliques

$$V_1 = \{0, 2, 4, 6, \dots, 2p^k - 2\} \text{ and } V_2 = \{1, 3, 5, 7, \dots, 2p^k - 1\}.$$

We observe that each clique of cardinality p^k different from V_1 and V_2 has non-empty intersection with V_1 and V_2.

We first study the particular cases for which $f_{p^k-1} = 2$. Suppose $1 \notin S$. In a clique V of p^k vertices different from V_1 and V_2, we must have p^k intervals between two consecutive vertices in V containing at least 1 vertex not in V, except for one containing 2 vertices, otherwise V could be identified with V_1 or V_2. It follows $|V(G) \setminus V| \geq p^k - 1 + 2 = p^k + 1$, that yields $|V| < p^k$. It contradicts the assumption.

Now suppose that $1 \in S$ and there exists t odd and coprime with p, $3 \leq t < p^k$, such that $t \notin S$. We prove $f_{p^k-1} = 2$. Towards a contradiction, let V be a clique of cardinality p^k different from V_1 and V_2, containing 0 and 1. Let $V' = \{v + t : v \in V\}$. If $V \cap V' \neq \varnothing$ there exist $v, w \in V$ such that $w = v + t$. This is impossible. If $V \cap V' = \varnothing$, then $V(G) = V \sqcup V'$. Since $(t, p) = 1$ and t odd, then $(t, n) = 1$, hence there exists an odd $a \in \mathbb{Z}_n$, coprime with n, such that $at \equiv 1 \mod n$. Since $0 \in V$ and $t \in V'$ by definition of V' we have that $2t \in V$. In fact if $t + t = 2t \in V'$, then $t \in V$, obtaining a contradiction. It follows that

$$2bt \in V \text{ and } (2b+1)t \in V' \text{ for any } b.$$

The vertex at lives in V' since a is odd and lives in V since $at = 1 \in V$. This is a contradiction.

Hence a clique of p^k vertices different from V_1 and V_2 cannot exist and $f_{p^k-1} = 2$.

We assume $f_{p^k-1} > 2$. Then by the previous observations we have

$$\left\{t : t \text{ odd and } (t, p) = 1\right\} \cup \left\{2k : k = 1, \dots, \frac{p^k-1}{2}\right\} \subseteq S.$$

Now we distinguish two cases:

(1) S is the $(p^k - 1)$-th power cycle, namely $S = \{1, \dots, p^k - 1\}$;
(2) S is not the $(p^k - 1)$-th power cycle.

(1) In this case f_{p^k-1} is the coefficient of the degree p^k term of the independence polynomial of the graph $C_n(p^k)$. As pointed out after Definition 3.4 in [2], this polynomial is

$$(1 + 2x)^{p^k}.$$

Hence $f_{p^k-1} = 2^{p^k} \equiv_p 2$ by Fermat's Little Theorem.

(2) If $S \neq \{1, \dots, p^k - 1\}$, there exists an odd multiple of p, mp for an odd m with $mp < p^k$, such that $mp \notin S$. Let $m = qp^r$ for some odd q with $\gcd(q, p) = 1$ and $0 \leq r < k - 1$. Let V be a clique of cardinality p^k different from V_1 and V_2. Let $V' = V + mp := \{v + mp : v \in V\}$. We have $V \cap V' = \varnothing$. Moreover if $v \in V$ then $v + mp \in V'$ and $v + 2mp \in V$. In fact if $v + 2mp \in V'$ then $mp \in S$ since V' is a clique. The latter implies that $V = V + 2mp = V + 2qp^{r+1}$. Since q is odd and coprime with p, it is coprime with n, hence it is invertible in $\mathbb{Z}_n$. Therefore, there exists $h \in \mathbb{Z}_n$ such that $qh \equiv_n 1$, and $2qhp^{r+1} \equiv_n 2p^{r+1}$. Since $V = V + 2qp^{r+1}$, then $V = V + 2p^{r+1}$. Now we prove that if $j \in \mathbb{Z}_n$ is such that

$$V = V + j,$$

then $2p \mid j$. Towards a contradiction, assume $V = V + j$ and $2p \nmid j$. We write $j = 2p^{r+1}a + b$ with $0 < b < 2p^{r+1}$ and $2p \nmid b$. Since $V = V + j$, then $V = V + b$ and we have

$$L = \{0, b, 2b, \dots (o(b) - 1)b\} \subseteq V$$

where $o(b)$ is the order of b in $(\mathbb{Z}_n, +)$. We analyse $g = \gcd(b, 2p^{r+1})$ to determine the order of b in $\mathbb{Z}_n$. Since $2p \nmid b$ and $b < 2p^{r+1}$, g could be either $1, 2, p^i$ with $1 \leq i \leq r + 1$.

If $g = 1$, then $o(b) = n$ and $L = \mathbb{Z}_n$, but $|V| = \frac{n}{2}$. This is impossible.

If $g = 2$, then $o(b) = p^k$, $L = V_1 \subseteq V$ and $|V_1| = |V|$ hence $V = V_1$. It is a contradiction to the assumption $V \neq V_1$.

If $g = p^i$, then $V = V + p^i = V + qp^{r+1-i}p^i = V'$. It contradicts the fact $V \cap V' = \varnothing$.

Hence, if $2p \nmid j$, then $V \neq V + j$. Let s be the minimum positive integer such that $V = V + 2sp$. Since $V = V + 2p^{r+1}$, it follows that $s \leq p^r$, $2p \leq 2sp < 2p^k$, and

$$V, V + 1, \dots, V + (2sp - 1)$$

are $2sp$ different cliques of G having cardinality p^k. Hence $2p$ divides $(f_{p^k-1} - 2)$ and

$$f_{p^k-1} \equiv_p 2.$$

The assertion follows. □

Example 2.8. We provide an example with $f_{p^k-1} > 2$ and $S \neq \{1, 2, \dots, p^k - 1\}$. We consider the graph $G = C_{50}(\{1, 2, \dots, 24\} \setminus \{5\})$, using the notation of the proof of Proposition 2.7, $V = V + 2mp = V + 2 \cdot 1 \cdot 5 = V + 10$. The clique complex of G has 32 cliques of cardinality 25. We fix a vertex v, for simplicity 0, and we look at the sequence of vertices $0, 1, \dots, 9$. Moreover, the symbol 0 denotes a vertex not in a clique, while the symbol 1 refers to a vertex in a clique. We have that V_1 has fundamental pattern 1010101010, while V_2 has fundamental pattern 0101010101. Since $V = V + 10$, each fundamental pattern is repeated 5 times to cover all the vertices of the graph. For example,

$$V_1 = 1010101010.1010101010.1010101010.1010101010.1010101010$$

and it happens for all the other cliques. The complex has 30 further cliques of three kinds, namely there are three further different patterns in the sequence of vertices in or not in a clique. The other three fundamental patterns are 1111100000, 1110100010 and 1101100100. Since the graph is circulant, for each of the last three patterns, there are 10 different cliques corresponding to the 10 possible shifts. For example, for the first sequence we will have

$$0111110000,\ 0011111000, \dots, 1111000001.$$

So the total number of cliques will be $3 \cdot 10 + 2 = 32$.

Now we are able to prove

Theorem 2.9. *Let p be an odd prime and let G be a non-empty circulant graph on $n = 2p^k$ vertices with $k > 0$. Then $\widetilde{\chi}(\Delta(G)) \neq 0$.*

Proof. By using similar arguments to Theorem 2.3 we say that

$$p \mid f_{i-1} \text{ with } 1 \leq i \leq p^k - 1.$$

So we write

$$\widetilde{\chi}(\Delta(G)) = pt + f_{p^k-1} - 1 \quad \text{for some } t \text{ in } \mathbb{Z}.$$

Since by Proposition 2.7 f_{p^k-1} is 0 or it is congruent 2 modulo p, we have

$$\widetilde{\chi}(\Delta(G)) = pr \pm 1$$

for some r in $\mathbb{Z}$. That is $\widetilde{\chi}(\Delta(G))$ does not vanish. □

Example 2.10. In the proofs of Theorems 2.3 and 2.9 we are giving a partial positive answer to the Conjecture 5.38 of [7] stating that all non-empty circulant graphs G have $\widetilde{\chi}(\Delta(G)) \neq 0$. But with a *MAGMA* algorithm, available at
http://www.giancarlorinaldo.it/eulercirculants.html,
we have found for $n = 30$ and $n = 36$ a list of circulants, up to isomorphisms, that contradict the Conjecture (see Table 1). Among those, for example, we report the circulant $G = C_{30}(\{1, 3, 8\})$ whose independence complex has f-vector equals [1, 30, 345, 1990, 6360, 11736, 12600, 7680, 2430, 300]. That is

$$\widetilde{\chi}(\Delta(G)) = -1 + 30 - 345 + 1990 - 6360 + 11736 - 12600 + 7680 - 2430 + 300 = 0.$$

We now give some applications. The structure and roots of the independence polynomial have been studied by Hoshino and Brown (see [1,7,2]). By Theorems 2.3 and 2.9, we obtain the following

Corollary 2.11. *Let $n \in \{p^k, 2p^k\}$ for a prime p and for $k > 0$, and let G be a non-empty circulant graph on n vertices. Then*

$$I(G, -1) \neq 0.$$

By Example 2.10 and Eq. (1.2), -1 is a root of the independence polynomial of the circulant graph $C_{30}(1, 3, 8)$.
Similar results follow by Remark 1.2 for the regularity index and the a-invariant. Moreover by using Corollary 4.8 of [6], we get the following result

Corollary 2.12. *Let G be a circulant graph as in Theorems 2.3 and 2.9. If G is Cohen–Macaulay then*

$$\operatorname{depth} R/I(G) = \operatorname{reg} R/I(G).$$

It is of interest to find other sufficient conditions under which the reduced Euler characteristic of a circulant graph does not vanish.

We focused on the number of vertices of a circulant graph, but other nice combinatorial properties like well-coveredness (see [7,5]), strongly connectedness (see [10]), vertex decomposability and shellability (see [12]) could be helpful. From another point of view, it would be nice to find entire classes of circulants that for particular n and S have vanishing Euler characteristic by using a theoretical approach, rather than the computational one used in Example 2.10.

Table 1
The table shows $G = C_n(S)$ such that $\widetilde{\chi}(\Delta(G)) = 0$, up to isomorphisms.

	$n = 30$	
{1, 3, 8}	{1, 7, 9, 11, 14}	{1, 2, 3, 7, 9, 11, 13}
{2, 9, 13}	{1, 4, 9, 13, 14}	{2, 3, 4, 5, 7, 9, 14}
{8, 9, 13}	{2, 3, 7, 8, 9}	{2, 3, 4, 5, 8, 9, 14}
{1, 8, 9, 14}	{1, 3, 4, 9, 11}	{1, 3, 4, 5, 7, 8, 14}
{2, 3, 11, 13}	{2, 7, 8, 9, 13}	{2, 3, 4, 5, 8, 11, 13}
{3, 8, 11, 13}	{2, 3, 4, 7, 13}	{1, 2, 3, 7, 8, 9, 11, 13}
{1, 3, 4, 13}	{1, 3, 4, 5, 7, 8}	{2, 3, 5, 8, 9, 11, 13, 14}
{7, 8, 9, 13}	{2, 3, 4, 5, 8, 11}	{1, 2, 3, 4, 5, 8, 9, 14}
{1, 4, 7, 9}	{1, 2, 3, 8, 9, 11}	{1, 2, 3, 5, 7, 9, 11, 14}
{1, 8, 9, 11}	{1, 3, 4, 7, 9, 13}	{2, 3, 4, 5, 7, 9, 13, 14}
{2, 9, 11, 14}	{1, 4, 7, 9, 11, 14}	{1, 3, 4, 5, 7, 8, 9, 11, 13}
{1, 2, 9, 13}	{1, 2, 3, 5, 11, 14}	{1, 2, 3, 4, 5, 8, 9, 11, 13}
{2, 3, 7, 9}	{1, 3, 4, 9, 11, 14}	{2, 3, 4, 5, 7, 8, 9, 13, 14}
{1, 7, 8, 9, 11}	{2, 3, 4, 7, 8, 13}	{1, 2, 3, 4, 5, 7, 9, 11, 13, 14}
{1, 3, 7, 8, 13}	{1, 2, 5, 7, 9, 13, 14}	
{2, 3, 4, 7, 8}	{1, 4, 5, 7, 8, 9, 11}	
	$n = 36$	
{2, 3, 6, 7, 10, 14, 15}	{2, 5, 6, 7, 10, 11, 14}	{2, 5, 6, 10, 11, 13, 14}
{1, 2, 5, 6, 7, 10, 11, 17}	{1, 5, 6, 7, 11, 13, 14, 17}	{2, 5, 6, 7, 10, 14, 15, 17}
{1, 2, 5, 6, 7, 10, 11, 13}	{1, 5, 6, 7, 10, 11, 13, 14, 17}	

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