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Chordal circulant graphs and induced matching number

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ABSTRACT

Let $G = C_n(S)$ be a circulant graph on n vertices. In this paper we characterize chordal circulant graphs and then we compute $\nu(G)$, the induced matching number of G. The latter are useful in bounding the Castelnuovo–Mumford regularity of the edge ideal $I(G)$ of G.

0. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{C}$ be a cycle of G. An edge $\{v, w\}$ in $E(G) \setminus E(\mathcal{C})$ with v, w in $V(\mathcal{C})$ is a *chord* of $\mathcal{C}$. A graph G is said to be *chordal* if every cycle has a chord.

We recall that a circulant graph is defined as follows. Let $S \subseteq T := \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $G := C_n(S)$ is a simple graph with $V(G) = \mathbb{Z}_n = \{0, \dots, n-1\}$ and $E(G) := \{\{i, j\} \mid |j - i|_n \in S\}$ where $|k|_n = \min\{|k|, n - |k|\}$. Given $i, j \in V(G)$ we call *labelling distance* the number $|i - j|_n$. By abuse of notation we write $C_n(a_1, a_2, \dots, a_s)$ instead of $C_n(\{a_1, a_2, \dots, a_s\})$.

Circulant graphs have been studied under combinatorial [2,3] and algebraic [8] points of view. In the former, the authors studied some families of circulants, i.e. the dth powers of a cycle, namely the circulants $C_n(1, 2, \dots, d)$ (that we will analyse in Section 3) and their complements. In the latter, the author studied some properties of the edge ideal of circulants. Let $R = K[x_0, \dots, x_{n-1}]$ be the polynomial ring on n variables over a field K. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Some algebraic properties and invariants of $R/I(G)$ can be derived from combinatorial properties of G. Chordality and the induced matching number have been used to give bounds on the Castelnuovo–Mumford regularity of $R/I(G)$ (see Section 1).

In Section 2 we prove that a circulant graph is chordal if and only if it is either complete or a disjoint union of complete graphs.

In Section 3 we give an explicit formula for the induced matching number of a circulant graph $C_n(S)$ depending on the cardinality and the structure of the set S. Moreover, by using `Macaulay2`, we compare the Castelnuovo–Mumford regularity of $R/I(G)$ with $\nu(G)$, the lower bound of Theorem 1.3, when G is the dth power of a cycle and n is less than or equal to 15. We report the result in Table 1.

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Table 1
The behaviour of reg $R/I(G)$ with respect to $\nu(G)$ for $G = C_n^d$.

G	$\nu(G)$	reg $R/I(G)$	G	$\nu(G)$	reg $R/I(G)$
$C_6(\{1\})$	2	2	$C_{12}(\{1,2,3\})$	2	2
$C_6(\{1,2\})$	1	1	$C_{12}(\{1,2,3,4\})$	2	2
$C_7(\{1\})$	2	2	$C_{12}(\{1,2,3,4,5\})$	1	1
$C_7(\{1,2\})$	1	2	$C_{13}(\{1\})$	4	4
$C_8(\{1\})$	2	3	$C_{13}(\{1,2\})$	3	3
$C_8(\{1,2\})$	2	2	$C_{13}(\{1,2,3\})$	2	2
$C_8(\{1,2,3\})$	1	1	$C_{13}(\{1,2,3,4\})$	2	2
$C_9(\{1\})$	3	3	$C_{13}(\{1,2,3,4,5\})$	1	2
$C_9(\{1,2\})$	2	2	$C_{14}(\{1\})$	4	5
$C_9(\{1,2,3\})$	1	2	$C_{14}(\{1,2\})$	3	3
$C_{10}(\{1\})$	3	3	$C_{14}(\{1,2,3\})$	2	2
$C_{10}(\{1,2\})$	2	2	$C_{14}(\{1,2,3,4\})$	2	2
$C_{10}(\{1,2,3\})$	2	2	$C_{14}(\{1,2,3,4,5\})$	2	2
$C_{10}(\{1,2,3,4\})$	1	1	$C_{14}(\{1,2,3,4,5,6\})$	1	1
$C_{11}(\{1\})$	3	4	$C_{15}(\{1\})$	5	5
$C_{11}(\{1,2\})$	2	2	$C_{15}(\{1,2\})$	3	3
$C_{11}(\{1,2,3\})$	2	2	$C_{15}(\{1,2,3\})$	3	3
$C_{11}(\{1,2,3,4\})$	1	2	$C_{15}(\{1,2,3,4\})$	2	2
$C_{12}(\{1\})$	4	4	$C_{15}(\{1,2,3,4,5\})$	2	2
$C_{12}(\{1,2\})$	3	3	$C_{15}(\{1,2,3,4,5,6\})$	1	2

1. Preliminaries

In this section we recall some concepts and notation that we will use later on in this article.

We recall that the circulant graph $C_n(1, 2, \ldots, \lfloor \frac{n}{2} \rfloor)$ is the complete graph K_n. Moreover, we compute the number of components of a circulant graph with the following

Lemma 1.1. *Let $S = \{a_1, \ldots, a_r\}$ be a subset of T and let $G = C_n(S)$ be a circulant graph. Then G has $\gcd(n, a_1, \ldots, a_r)$ disjoint components. In particular, G is connected if and only if $\gcd(n, a_1, \ldots, a_r) = 1$.*

For a proof see [1]. From Lemma 1.1 it follows that if $n = dk$, then the disjoint components of $C_n(a_1d, a_2d, \ldots, a_sd)$ are d copies of the circulant graph $C_k(a_1, a_2, \ldots, a_s)$.

Let G be a graph. A collection C of edges in G is called an *induced matching* of G if the edges of C are pairwise disjoint and the graph having C has edge set is an induced subgraph of G. The maximum size of an induced matching of G is called *induced matching number* of G and we denote it by $\nu(G)$.

Let $\mathbb{F}$ be the minimal free resolution of $R/I(G)$. Then

$$\mathbb{F} \ : \ 0 \to F_p \to F_{p-1} \to \ldots \to F_0 \to R/I(G) \to 0$$

where $F_i = \bigoplus_j R(-j)^{\beta_{i,j}}$. The $\beta_{i,j}$ are called the *Betti numbers* of $\mathbb{F}$. The *Castelnuovo–Mumford regularity* of $R/I(G)$, denoted by reg $R/I(G)$ is defined as

$$\text{reg } R/I(G) = \max\{j - i : \beta_{i,j} \neq 0\}.$$

Let G be a graph. The *complement graph* $\bar{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the non-edges of G. We conclude the section by stating some known results relating chordality and induced matching number to the Castelnuovo–Mumford regularity. The first one is due to Fröberg ([4, Theorem 1]).

Theorem 1.2. *Let G be a graph. Then $\operatorname{reg} R/I(G) \leq 1$ if and only if $\bar{G}$ is chordal.*

The second one is due to Katzman ([7, Lemma 2.2]).

Theorem 1.3. *For any graph G, we have $\operatorname{reg} R/I(G) \geq \nu(G)$.*

When G is the circulant graph $C_n(1)$, namely the cycle on n vertices, we have the following result due to Jacques [6].

Theorem 1.4. *Let C_n be the n-cycle and let $I = I(C_n)$ be its edge ideal. Let $\nu = \lfloor \frac{n}{3} \rfloor$ denote the induced matching number of C_n. Then*

$$\operatorname{reg} R/I = \begin{cases} \nu & \text{if } n \equiv 0, 1 \pmod 3 \\ \nu + 1 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

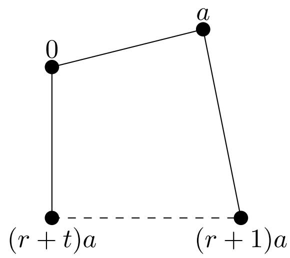

Fig. 1. Some edges of a non-chordal cycle of G.

2. Chordality of circulants

The aim of this section is to prove the following

Theorem 2.1. *Let G be a circulant graph. Then G is chordal if and only if there exists $d \geq 1$ such that $n = dm$ and $G = C_n(d, 2d, \dots, \lfloor \frac{m}{2} \rfloor d)$.*

The ($\Leftarrow$) implication is trivial. If $d = 1$, then G is the complete graph K_n, while if $d > 1$, then G is the disjoint union of d complete graphs K_m.

To prove ($\Rightarrow$) implication we need some preliminary results.

Lemma 2.2. *Let $G = C_n(S)$ be a circulant graph. Let us assume that there exists $a \in S$ with $k = \mathrm{ord}(a) \geq 4$ such that*

$$\left\{a, 2a, \dots, \left\lfloor \frac{k}{2} \right\rfloor a\right\} \not\subseteq S.$$

Then G is not chordal.

Proof. Since $k \geq 4$, then $\{a\} \subset \{a, 2a, \dots, \lfloor \frac{k}{2} \rfloor a\}$. If $\left\{a, 2a, \dots, \left\lfloor \frac{k}{2} \right\rfloor a\right\} \not\subseteq S$, then we have two cases:

$(1S)$ $\{a, 2a, \dots, ra, (r+t)a\} \subseteq S$ and $(r+1)a, \dots, (r+t-1)a \notin S$, with $r \geq 1$ and $t \geq 2$;
$(2S)$ $\{a, 2a, \dots, ra\} \subseteq S$ and $(r+1)a, \dots, \lfloor \frac{k}{2} \rfloor a \notin S$, with $1 \leq r < \lfloor \frac{k}{2} \rfloor$.

$(1S)$ We want to find a non-chordal cycle of G. We consider the edges $\{0, (r+t)a\}$, $\{0, a\}$, $\{a, (r+1)a\}$ (see Fig. 1). If $(r+1)a$ is adjacent to $(r+t)a$, then we found a non-chordal cycle of G. Otherwise, we apply the division algorithm to $r+t$ and $r+1$, that is

$$r + t = (r+1)q + s \quad 0 \leq s \leq r.$$

From the vertex $(r+1)a$ we alternately add a and ra to get the multiples of $(r+1)a$, until $q(r+1)a$. If $s = 0$, then we get $(r+t)a$, otherwise $0 < s \leq r$ and $sa \in S$ so we join $q(r+1)a$ and $(r+t)a$. The above cycle has length greater than or equal to 4 because the vertices $0, a, (r+1)a, (r+t)a$ are different. Furthermore, it is non-chordal because by construction any pair of non-adjacent vertices in the cycle has labelling distance in $\{(r+1)a, \dots, (r+t-1)a\}$.

$(2S)$ As in case $(1S)$, we want to construct a non-chordal cycle of G. We write $k = \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil$ and $\lfloor \frac{k}{2} \rfloor = qr + t$ with $0 \leq t \leq r - 1$. Now we write $\lceil \frac{k}{2} \rceil = qr + s$, where

$$s = \begin{cases} t & \text{if } k \text{ even} \\ t+1 & \text{if } k \text{ odd,} \end{cases}$$

and we take the cycle on vertices

$$\left\{0, ra, 2ra, \dots, qra, \left\lfloor \frac{k}{2} \right\rfloor a, \left\lfloor \frac{k}{2} \right\rfloor a + ra, \left\lfloor \frac{k}{2} \right\rfloor + 2ra, \dots \left\lfloor \frac{k}{2} \right\rfloor a + qra\right\}. \tag{2.1}$$

Since $r < \lfloor \frac{k}{2} \rfloor$, then $q \geq 1$ and in the case $q = 1$, $s > 0$. That is, the cycle on vertices (2.1) has length at least 4 and it is not chordal because by construction any pair of non-adjacent vertices in the cycle has labelling distance in $\{(r+1)a, \dots, \lfloor \frac{k}{2} \rfloor a\}$.

In any case G is not chordal and the assertion follows. □

An immediate consequence of the previous lemma is

Corollary 2.3. *Let $G = C_n(S)$ be a circulant graph. If there exists $a \in S$ with $k = \mathrm{ord}(a) \geq 4$ such that $\gcd(a, n) \notin S$, then G is not chordal.*

Lemma 2.4. *Let $G = C_n(S)$ be a circulant graph. If $a_1, \dots, a_r \in S$ and $\gcd(a_1, \dots, a_r) \notin S$, then G is not chordal.*

Proof. We proceed by induction on r.

Let $r = 2$ and let $a_1, a_2 \in S$ be such that $c = \gcd(a_1, a_2) \notin S$. We consider

$$a = \gcd(a_1, n), \;\; b = \gcd(a_2, n), \;\; d = \gcd(a, b).$$

From Corollary 2.3, we have that if one between a, b does not belong to S, then G is not chordal. Hence $a, b \in S$. We have that d divides c and we distinguish two cases. If $d \in S$, since $c = td \notin S$ for some t, then by Lemma 2.2 G is not chordal. Therefore, from now on we suppose $d \notin S$. Since a and b divide n, then $\mathrm{lcm}(a, b) = \frac{ab}{d}$ divides n. We want to find a non-chordal cycle of G having length 4. Let $ra + sb = d \pmod n$ be a Bézout identity of a and b. From Lemma 2.2, if one between ra and sb is not in S, then G is not chordal. Hence, let us assume $ra, sb \in S$. Now we consider the cycle

$$\{0, ra, ra + sb = d, sb\}.$$

Since $d \notin S$, then the edge $\{0, d\} \notin E(G)$. We distinguish two cases about $ra - sb$. If $ra - sb \notin S$, then the assertion follows. If $ra - sb \in S$, then we set

$$kd = \gcd(ra - sb, n) \Rightarrow k = \gcd\left(r\left(\frac{a}{d}\right) + s\left(\frac{b}{d}\right), \frac{n}{d}\right).$$

If kd is not in S, then from Corollary 2.3 G is not chordal. Hence, we consider $kd \in S$. Since $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$, then $\gcd\left(k, \frac{a}{d}\right) = \gcd\left(k, \frac{b}{d}\right) = 1$, and

$$\gcd\left(k, \frac{ab}{d^2}\right) = 1 \;\Rightarrow \gcd\left(kd, \frac{ab}{d}\right) = d. \tag{2.2}$$

Hence $\mathrm{lcm}\left(kd, \frac{ab}{d}\right) = k\frac{ab}{d}$ divides n. We distinguish two cases. If $k = 1$, then we obtain the contradiction $d \in S$, arising from the assumption $ra - sb \in S$. If $k \neq 1$, then k is a new proper divisor of n. We set $a' = kd$ and $b' = \frac{ab}{d}$, we apply the steps above and we find a k' so that $k'\frac{a'b'}{d}$ divides n, and so on. By applying the steps above to a' and b' a finite number of times, we could either find a k' equal to 1 or we could get new proper divisors of n, that are finite in number. We want to study the case $n = \frac{a'b'}{d}$. Let

$$va' + zb' = d$$

be a Bézout identity, we assume $va' - zb' \in S$, and we set

$$hd = \gcd\left(va' + zb', n\right).$$

We have that $h\frac{a'b'}{d} = hn$ divides n, that is $hn = n$ and $h = 1$. It implies $d \in S$, that is a contradiction arising from the assumption $va' - zb' \in S$. Hence $va' - zb' \notin S$ and $\{0, va', d, zb'\}$ is a non-chordal cycle of G. It ends the induction basis. For the inductive step, we suppose the statement true for $r - 1$ and we prove it for r. We have to prove that if $\gcd(a_1, \dots, a_r) \notin S$, then G is not chordal. By inductive hypothesis if $\gcd(a_1, \dots, a_{r-1}) \notin S$, then G will be not chordal. Hence we assume $b = \gcd(a_1, \dots, a_{r-1}) \in S$. By applying the inductive basis to a_r and b, we obtain that G is not chordal. □

Now we are able to complete the proof of Theorem 2.1.

Proof of Theorem 2.1 (⇒). Under the hypothesis that G is chordal, we also assume that G is connected and we prove that $d = 1$, that is $G = K_n$. By contradiction assume that the graph is not complete, namely $G = C_n(a_1, \dots, a_s)$ with $s < \lfloor \frac{n}{2} \rfloor$. From Lemma 1.1, G is connected if and only if $\gcd(a_1, \dots, a_s, n) = 1$. Let $b = \gcd(a_1, \dots, a_s)$.

If $b \notin S$, then from Lemma 2.4 G is not chordal. If $b \in S$, we have $1 = \gcd(n, a_1, \dots, a_s) = \gcd(n, \gcd(a_1, \dots, a_s)) = \gcd(n, b)$. If $1 \notin S$, then from Lemma 2.4, G is not chordal. Then $1 \in S$ and from Lemma 2.2 the graph G is not chordal, that is a contradiction. If G is not connected, then it has $a = \gcd(n, S)$ distinct components, each of $m = \mathrm{ord}(a)$ vertices. By Lemma 2.2, $S = \{a, 2a, \dots, \lfloor \frac{m}{2} \rfloor a\}$ and each component is the complete graph K_m. □

Example 2.5. Here we present three examples of non-chordal circulant graphs $C_n(S)$.

1. Take $n = 15$ and $S = \{2, 3, 4, 7\}$. If we take $a = 2$, then $\mathrm{ord}(a) = 15$ and $2a = 4$, $3a = 6$, $n - 4a = 7$, and $n - 6a = 3$. Hence, we are in case (1S) of Lemma 2.2 with $S = \{a, 2a, 4a, 6a\}$. We observe that the cycle on vertices

$$\{0, a, 3a, 4a\} = \{0, 2, 6, 8\}$$

is not chordal because $6 \notin S$.

2. Take $n = 10$, $S = \{3, 4\}$ and $a = 3$. We have $\mathrm{ord}(a) = 10$. Moreover $n - 2a = 4$, hence this is the case (2S) of Lemma 2.2 with $S = \{a, 2a\}$. We have $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = 5$, and
$$5 = qr + t = 2 \cdot 2 + 1.$$
Hence, we take the cycle on vertices
$$\{0, 2a, 4a, 5a, 7a, 9a\} = \{0, 6, 2, 5, 1, 7\}$$
that is not chordal because 1, 2 and 5 do not belong to S.
3. We take $n = 30$ and $S = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15\}$. We observe that $\gcd(5, 2) = 1 \notin S$, hence we are in the case of Lemma 2.4 with $a_1 = a = 5$ and $a_2 = b = 2$. We observe that $\mathrm{ord}(a) = 6, ord(b) = 15$ and $2a = 10, 3a = 15, b, 2b, \ldots, 7b \in S$. We take a Bezóut identity of a and b
$$1 = ra + sb = 5 \cdot 1 - 2 \cdot 2.$$
We take the cycle on vertices $\{0, 5, 1, -4\}$. The quantity $ra - sb = 5 + 4 = 9$ belongs to S and $k = \gcd(9, 30) = 3$, while $\gcd(k, ab) = \gcd(3, 10) = 1$ and $n = abk = 30$. Hence we write
$$1 = vab + sk = 10 - 3 \cdot 3,$$
and we take the cycle on vertices $\{0, 10, 1, -9\}$. The quantity $10 + 9 = 19$ does not belong to S, hence the cycle above is not chordal.

3. Induced matching number of circulant graphs

In this section we compute the induced matching number for any circulant graph $C_n(S)$. Then we plot a table representing the behaviour of $\mathrm{reg}\, R/I(G)$ with respect to the lower bound described in Theorem 1.3, when G is the dth power of the cycle, namely $G = C_n(1, 2, \ldots, d)$. For the computation we used `Macaulay2` (see [5]).

Definition 3.1. Let G be a graph with edge set $E(G)$. We say that two edges e, e' are *adjacent* if $e \cap e' = v$ and $v \in V(G)$. We say that e, e' are 2-*adjacent* if there exist $v \in e$ and $u \in e'$ such that $\{u, v\} \in E(G)$.

Remark 3.2. From Definition 3.1, an induced matching of G is a subset of $E(G)$ where the edges are not pairwise adjacent or 2-adjacent.

Then we have the following

Theorem 3.3. *Let $G = C_n(S)$ be a connected circulant graph, let $s = |S|$ and let $r = \min S$. Then $\nu(G) = \lfloor \frac{|E(G)|}{t} \rfloor$ where*
$$t = \begin{cases} s^2 + (|A| + 1)s & \text{if } \frac{n}{2} \notin S \\ s^2 + (|A| + 1)s - 2 & \text{if } \frac{n}{2} \in S, \end{cases}$$
with
$$A = \Big\{ r + a \ : \ a \in S \ \text{ and } \ r + a \in V(G) \setminus S \Big\}.$$
If G has $d = gcd(n, S)$ components, then $\nu(G) = d \cdot \nu(C_{n/d}(S'))$, where $S' = \{s/d \ : \ s \in S\}$.

Proof. We consider some disjoint subsets of $E(G)$, E_i $i = 1, \ldots, m$ consisting in an edge $e_i = \{u, v = u + s\}$ for an $s \in S$, the edges $\{v, w = v + s\}$ for an $s \in S$ adjacent to e_i, and the edges $\{w, w + s\}$ for an $s \in S$ 2-adjacent to e_i. By suitably choosing the e_i, the $\{e_i\}_{i=1,\ldots,m}$ is the biggest induced matching and $m = \nu(G)$. So we have only to count the edges in any set E_i.
We assume that $s = |S|$, $r = \min S$ and $S = \{a_0 = r, a_1, \ldots, a_{s-1}\}$, we assume that the edge $e = \{0, r\}$ is in the induced matching, and let E' be the set containing e and the edges adjacent or 2-adjacent to e. The edges adjacent to e are $\{0, a_i\}$ for $i = 1, \ldots, s - 1$ and $\{r, b_i = r + a_i\}$ for $i = 0, \ldots, s - 1$. The above edges are all distinct. The edges 2-adjacent to e are $\{a_j, a_j + a_i\}$ for $j \in \{1, \ldots, s - 1\}, i \in \{0, \ldots, s - 1\}$ and $\{b_j, b_j + a_i\}$ for $i, j \in \{0, \ldots, s - 1\}$. The edges above may not be all distinct. In fact, it can happen that some b_j coincides with some a_k, in that case $\{b_j, b_j + a_i\} = \{a_k, a_k + a_i\}$ for any $i \in \{0, \ldots, s - 1\}$. Then, we only consider $\{b_j, b_j + a_i\}$ for $i \in \{0, \ldots, s - 1\}$ when $b_j \in A$. To sum up, in the set E' we find:

(a) The s edges $\{0, a_i\}$ for $i \in \{0, \ldots, s - 1\}$;
(b) The s^2 edges $\{a_j, a_j + a_i\}$ for $i, j \in \{0, \ldots, s - 1\}$;
(c) The $s \cdot |A|$ edges $\{b, b + a_i\}$ for $i \in \{0, \ldots, s - 1\}$ and $b \in A$.

If $a_{s-1} = \frac{n}{2}$, then $b_{s-1} = r + a_{s-1} \in A$ and the edges $\{a_{s-1}, a_{s-1} + a_{s-1} = 0\}$ of point (b) and $\{b_{s-1}, b_{s-1} + a_{s-1} = r\}$ of point (c) are already counted. The assertion follows.

For the case disconnected, let $d = gcd(n, S)$ be the number of disjoint connected components of the graph G. Since the components are disjoint, it turns out that $\nu(G)$ is d times the induced matching number of one component. That component is $C_{n/d}(S')$ where $S' = \{s/d \ : \ s \in S\}$, hence the assertion follows. □

The formula in Theorem 3.3 can be written in a compact way when G is the dth power of a cycle. We set $C_n^d = C_n(\{1, 2, \ldots, d\})$.

Corollary 3.4. *Let $G = C_n^d$ be the dth power of a cycle and $d < \lfloor \frac{n}{2} \rfloor$. Then*

$$\nu(G) = \left\lfloor \frac{n}{d+2} \right\rfloor.$$

Proof. We want to apply Theorem 3.3, with $s = d$ and $|E(G)| = nd$. We have $r = 1$ and $A = \{d + 1\}$. Hence it follows that $t = d^2 + d + d \cdot 1 = d^2 + 2d = d(d + 2)$, that is

$$\nu(G) = \left\lfloor \frac{nd}{d(d+2)} \right\rfloor = \left\lfloor \frac{n}{d+2} \right\rfloor. \quad \square$$

In Table 1, we compare the values of $\operatorname{reg} R/I(C_n^d)$ for $n \leq 15$ and $1 \leq d \leq \lfloor \frac{n}{2} \rfloor$. We highlight that the regularity of $R/I(G)$ is strictly greater than $\nu(G)$ in two different cases:

(1) when $G = C_n$ and $n \equiv 2 \pmod 3$.
(2) when $G = C_n^{\lfloor \frac{n}{2} \rfloor - 1}$ and n is odd.

The two anomalous cases were expected: in case (1), we know from Theorem 1.4 that $\operatorname{reg} R/I(G) = \nu + 1$; in case (2), $\nu(G) = 1$ while $\bar{G} = C_n(\lfloor \frac{n}{2} \rfloor)$ that is a cycle and hence it is not chordal; hence from Theorem 1.2 we know that $\operatorname{reg} R/I(G) = 2$. In general, it seems that apart from cases (1) and (2), the Castelnuovo–Mumford regularity of the dth power of a cycle grips the bound of $\nu(G)$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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