



On nonlinear electromechanics of piezoelectric nanobeams[☆]

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ABSTRACT

Electromechanics of slender piezoelectric nanobeams is addressed by an effective nonlocal nonlinear methodology. Kinematics is modeled according to the Bernoulli-Euler beam theory. Geometric nonlinearities are captured exploiting the von Kármán approach. The constitutive law of piezoelectricity, coupling mechanical and electrostatic behaviors, is combined with a nonlocal formulation accounting for size effects. Notably, the stress-driven nonlocal theory of integral elasticity is adopted and reverted to an equivalent differential formulation for theoretical and computational purposes. The relevant structural problem of piezoelectric nonlocal elastic beams is derived and solved for benchmark case-studies. Parametric analyses are carried out to show the influence of scale effects and electric voltages on the electromechanical response. The proposed methodology can enhance modeling and design of new-generation nano-electromechanical systems.

1. Introduction

Nowadays, innovative nanomechanical applications involve piezoelectric materials due to their appealing features. Indeed, nanodevices based on piezoelectric nanobeams [1–4] offer high sensitivity for enhanced performance of nanosensors, provide combined sensing and actuation modes, operate with low power consumption, and ensure fast response times for converting mechanical signals into electrical ones and vice-versa. Because of their ability to link electrical and mechanical responses, piezoelectric structures (see e.g. [5,6]) are suitable for cutting-edge nanotechnological applications in which electromechanical coupling plays a pivotal role [7–9].

When dealing with advanced materials and small-scale structures, sophisticated tools are required to capture non-conventional responses (see e.g. [10–20]). An advanced theoretical framework is represented by nonlocal continuum mechanics, providing refined models essential to account for scale-dependent behaviors of nanostructures. The earliest ideas underlying nonlocal continuum formulations were introduced in the seminal works by Kröner [21], Krumhansl [22], Kunin [23], Rogula [24,25]. Later, Eringen [26,27] proposed an integral law of elasticity in which stress is given by spatial convolution driven by the elastic strain field. Nevertheless, difficulties emerged when applying Eringen's theory to nanomechanics, as first detected by Peddieson et al. [28] and then discussed by Benvenuti and Simone [29], Khodabakhshi and Reddy [30]. A comprehensive explanation was provided by Romano et al.

[31], who demonstrated that the loss of well-posedness in structural boundary-value problems governed by Eringen's model stems from incompatibility between equilibrium requirements and the strain-driven integral constitutive law.

Consequently, Romano and Barretta [32] proposed an alternative integral theory of nonlocal elasticity in which nonlocal elastic strains are spatial convolutions driven by the stress field. Such a constitutive law, which is not the inverse of the strain-driven Eringen's model, guarantees the well-posedness of the governing structural problem and accurately describes size effects arising from long-range interactions as confirmed by Jafarinezhad et al. [33], Lovisi et al. [34], Indronil [35], Behnam-Rasouli et al. [36], Challamel and Sani [37], Barretta et al. [38].

Electromechanics of piezoelectric nanobeams is here investigated. Geometric nonlinearity is introduced by exploiting the approach proposed by von Kármán [39]. The constitutive law of piezoelectricity is combined with the consistent nonlocal stress-driven model accounting for size effects. The proposed formulation extends previous contributions in scientific literature, which can be recovered as special cases by suppressing axial–transverse coupling (see e.g. [40,41]) or geometrically nonlinear effects (see e.g. [3]).

Here is the outline. In Section 2, kinematics is modeled according to the Bernoulli-Euler theory of slender beams and the von Kármán nonlinear formulation is introduced. The variational formulation of

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equilibrium is then exploited to derive the differential and boundary conditions of equilibrium. Nonlocal elasticity for nanobeams is described in Section 3, and an equivalent differential problem made of a second order differential equation equipped with non-standard boundary conditions is derived. The mathematical formulation of piezoelectricity applied to nanobeams is provided in Section 4. Then, the electromechanical problem of piezoelectric nonlocal elastic beams is formulated in Section 5. Parametric analyses are performed in Section 6 to investigate the effects of nonlocal parameter and electric voltage on electromechanical responses. Concluding remarks are provided in Section 7.

2. Kinematics and equilibrium

Let us consider a slender beam \mathcal{B} of length L and cross-section Ω , whose axis is described by the x coordinate, while the z coordinate is taken along the bending axis. Kinematics is formulated according to the Bernoulli-Euler theory of planar beams, so that the displacement field \mathbf{w} is represented by

$$|\mathbf{w}| = \begin{vmatrix} w_x \\ w_y \\ w_z \end{vmatrix} = \begin{vmatrix} u(x) - z \partial_x v(x) \\ 0 \\ v(x) \end{vmatrix}, \quad (1)$$

where u and v are the axial and transverse displacement fields of the beam axis and $\partial_x(\bullet)$ denotes derivative of \bullet along x . The geometric nonlinearities are taken into account through the von Kármán formulation, i.e.

$$\epsilon_x = \partial_x u(x) + \frac{1}{2} (\partial_x v(x))^2 - z \partial_x^2 v(x), \quad (2)$$

being ϵ_x the axial strain [42].

According to the variational formulation of equilibrium, the internal virtual work

$$\int_{\mathcal{B}} \sigma_x \delta \epsilon_x dV := \int_0^L \int_{\Omega} \sigma_x \delta \epsilon_x dA dx, \quad (3)$$

must be equal to the external virtual work for any virtual displacement fields δu and δv satisfying homogeneous kinematic boundary conditions. Integration by parts and localization lead to the following differential equations of equilibrium

$$\begin{cases} \partial_x N(x) = -p(x), \\ \partial_x^2 M(x) + \partial_x (N \cdot \partial_x v)(x) = -q(x), \end{cases} \quad (4)$$

equipped with boundary conditions

$$\begin{cases} N(0) \delta u(0) = -\mathcal{F}_{x,0} \delta u(0), \\ N(L) \delta u(L) = \mathcal{F}_{x,L} \delta u(L), \\ M(0) \partial_x \delta v(0) = \mathcal{M}_0 \partial_x \delta v(0), \\ M(L) \partial_x \delta v(L) = -\mathcal{M}_L \partial_x \delta v(L), \\ (N \partial_x v + \partial_x M)(0) \delta v(0) = -\mathcal{F}_{z,0} \delta v(0), \\ (N \partial_x v + \partial_x M)(L) \delta v(L) = \mathcal{F}_{z,L} \delta v(L), \end{cases} \quad (5)$$

being p and q axial and transverse distributed loadings and $F_{i,j}$, \mathcal{M}_j , with $i \in \{x, z\}$ and $j \in \{0, L\}$, concentrated forces and bending couples, respectively. It is worth noting that the following static equivalences have been defined

$$N := \int_{\Omega} \sigma_x dA, \quad M := \int_{\Omega} z \sigma_x dA, \quad (6)$$

where σ_x is the axial stress field. In Eq. (6), N and M stand for the axial and bending interaction fields, respectively.

3. Nonlocal elasticity for nanobeams

Size dependent mechanical behaviors of nanobeams are modeled by the stress-driven nonlocal theory [32] according to which the nonlocal

elastic curvature χ^{el} is given by the following constitutive integral law

$$\chi^{el}(x) = \int_0^L \phi_c(x-\eta) \left(\frac{M}{K_f} \right)(\eta) d\eta. \quad (7)$$

The latter equation expresses the nonlocal curvature field as spatial convolution between averaging function ϕ_c and bending interaction field M . Symbol K_f in Eq. (7) denotes the local elastic flexural stiffness. The kernel ϕ_c is described by a characteristic length $c \in]0, +\infty[$. As suggested by Eringen [27], the averaging kernel can be selected as the Helmholtz's bi-exponential function

$$\phi_c(x) = \frac{1}{2c} e^{-\frac{|x|}{c}}, \quad (8)$$

since it satisfies peculiar properties that enable inversion of the constitutive integral law Eq. (7) into the following differential equation

$$\chi(x) - c^2 \partial_x^2 \chi(x) = \left(\frac{M}{K_f} \right)(x), \quad (9)$$

equipped with the constitutive boundary conditions

$$\begin{cases} \partial_x \chi(0) = \frac{\chi(0)}{c}, \\ \partial_x \chi(L) = -\frac{\chi(L)}{c}, \end{cases} \quad (10)$$

as proven by [32]. In absence of nonelastic effects, coincidence of nonlocal elastic and geometric curvatures can be assumed, i.e. $\chi^{el} = \partial_x^2 v$. The equivalent differential problem provided by Eqs. (9)–(10) is theoretically and computationally convenient, as shown in the following.

4. Piezoelectricity for nanobeams

In the absence of free electric charges, Gauss theorem writes as

$$\text{div } \mathbf{D} = 0, \quad (11)$$

id est, the electric displacement \mathbf{D} of the piezoelectric nanobeam is a solenoidal field. Eq. (11) can be written in terms of components as follows

$$\partial_x D_x + \partial_z D_z = 0, \quad (12)$$

being D_x and D_z the components of the electric displacement along x and z axes, respectively, that are explicitly expressed by the constitutive laws of linear piezoelectricity [43], i.e.

$$\begin{cases} D_x = \lambda_x \Phi_x, \\ D_z = e_{zx} \epsilon_x + \lambda_z \Phi_z. \end{cases} \quad (13)$$

In Eq. (13), λ_x and λ_z are the dielectric constants, while e_{zx} is the piezoelectric coefficient; Φ_x and Φ_z are the components of the electric field Φ , which is related to the electric potential ψ by the following equation

$$\Phi = -\nabla \psi. \quad (14)$$

Assuming $\Phi_x \ll \Phi_z$, the electric displacement D_x can be neglected in comparison with D_z [44,45]. Gauss theorem in Eq. (12) thus prescribes uniformity of the electric displacement D_z along z , i.e. $\partial_z D_z = 0$, that can be further expressed by taking into account Eq. (13)₂, Eq. (2) and (14) to obtain

$$\partial_z^2 \psi(x, z) = -\frac{e_{zx}}{\lambda_z} \partial_x^2 v(x). \quad (15)$$

The electric potential ψ can be thus derived by solving the differential equation (15) equipped with the boundary conditions $\psi\left(x, -\frac{h}{2}\right) = 0$ and $\psi\left(x, \frac{h}{2}\right) = V$, being V the applied voltage and h the maximum

dimension of cross-section along the bending axis z . It is worth noting that the cross-section has been assumed to be doubly symmetric with respect to the y and z axes. Hence, the following expression is derived for the electric potential [46], i.e.

$$\psi(x, z) = -\frac{e_{zx}}{\lambda_z} \partial_x^2 v(x) \left(\frac{z^2}{2} - \frac{h^2}{8} \right) + V \left(\frac{1}{2} + \frac{z}{h} \right). \quad (16)$$

The constitutive law of linear piezoelectricity writes as

$$\sigma_x = E \varepsilon_x - e_{zx} \Phi_z. \quad (17)$$

By taking into account Eq. (2), (14) and (16), the constitutive law in Eq. (17) is given by

$$\sigma_x = E \partial_x u(x) + \frac{E}{2} (\partial_x v(x))^2 - E z \partial_x^2 v(x) - \frac{e_{zx}^2}{\lambda_z} \partial_x^2 v(x) z + e_{zx} \frac{V}{h}. \quad (18)$$

5. Electromechanical bending of piezoelectric nanobeams

The structural problem of piezoelectric nonlocal elastic beams is here derived. Considering the static equivalence Eq. (6)₁ and the expression of the axial stress provided by the constitutive law Eq. (18), the axial interaction field N is obtained

$$N = E A \partial_x u(x) + \frac{E A}{2} (\partial_x v)^2 + e_{zx} A \frac{V}{h}, \quad (19)$$

being A the cross sectional area.

Combining the static equivalence Eq. (6)₂ for the bending interaction field M with the constitutive law in Eq. (18) yields

$$M = -\left(\int_{\Omega} \left(E + \frac{e_{zx}^2}{\lambda_z} \right) z^2 dA \right) \partial_x^2 v. \quad (20)$$

Denoting with $\bar{K}_f := \int_{\Omega} \left(E + \frac{e_{zx}^2}{\lambda_z} \right) z^2 dA$ the effective flexural stiffening, Eq. (20) can be simply rewritten as $\partial_x^2 v = -\frac{M}{\bar{K}_f}$. Exploiting the equivalent differential constitutive equation (9), the bending interaction field M can be expressed as

$$M(x) = -\bar{K}_f (\partial_x^2 v(x) - c^2 \partial_x^4 v(x)). \quad (21)$$

Substituting Eqs. (19) and (20) into the differential equilibrium equations (4), the following set of coupled nonlinear differential equations is obtained

$$\begin{cases} E A (\partial_x^2 u + \partial_x v \partial_x^2 v) = -p, \\ -\bar{K}_f \partial_x^4 v + c^2 \bar{K}_f \partial_x^6 v + \frac{3}{2} E A (\partial_x v)^2 \partial_x^2 v + e_{zx} \frac{V}{h} A \partial_x^2 v \\ + E A (\partial_x v \partial_x^2 u + \partial_x u \partial_x^2 v) = -q, \end{cases} \quad (22)$$

governing electromechanical bending of piezoelectric nanobeams. Eq. (22) must be equipped with constitutive boundary conditions

$$\begin{cases} \partial_x^3 v(0) = \frac{\partial_x^2 v(0)}{c}, \\ \partial_x^3 v(L) = -\frac{\partial_x^2 v(L)}{c}, \end{cases} \quad (23)$$

and with standard boundary conditions Eq. (5).

It is convenient to introduce the following non-dimensional fields

$$\begin{aligned} \xi &:= \frac{x}{L}, & \lambda_c &:= \frac{c}{L}, & v^* &:= \frac{v}{L}, & u^* &:= \frac{u}{L}, & k^* &:= \frac{E A L^2}{\bar{K}_f}, \\ e_{zx}^* &:= \frac{A V e_{zx} L^2}{\bar{K}_f h}, & q^* &:= \frac{q L^3}{\bar{K}_f}, & p^* &:= \frac{p L}{E A}, \end{aligned} \quad (24)$$

so that the governing differential problem Eq. (22) can be rewritten in the following non-dimensional form:

$$\begin{cases} \partial_{\xi}^2 u^* + \partial_{\xi} v^* \partial_{\xi}^2 v^* = -p^*, \\ -\partial_{\xi}^4 v^* + \lambda_c \partial_{\xi}^6 v^* + \frac{3}{2} k^* (\partial_{\xi} v^*)^2 \partial_{\xi}^2 v^* + k^* (\partial_{\xi} v^* \partial_{\xi}^2 u^* \\ + \partial_{\xi} u^* \partial_{\xi}^2 v^*) + e_{zx}^* \partial_{\xi}^2 v^* = -q^*. \end{cases} \quad (25)$$

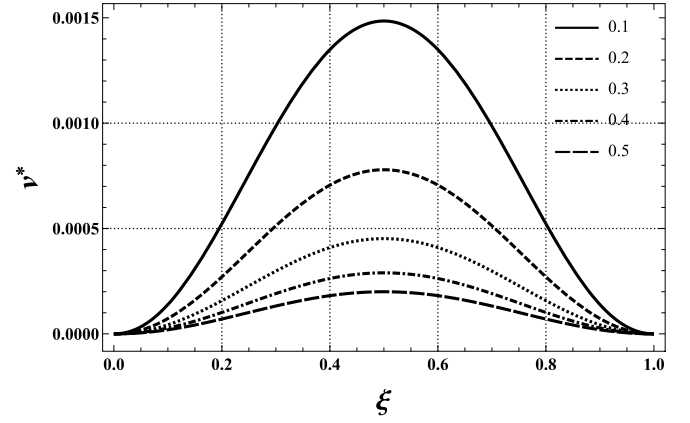


Fig. 1. Fixed piezoelectric nanobeam: non-dimensional displacement field v^* for increasing nonlocal parameter λ_c , for an applied voltage $V = 0.1$ V.

Eq. (25) is equipped with non-dimensional constitutive boundary conditions

$$\begin{cases} \partial_{\xi}^3 v^*(0) = \frac{\partial_{\xi}^2 v^*(0)}{\lambda_c}, \\ \partial_{\xi}^3 v^*(L) = -\frac{\partial_{\xi}^2 v^*(L)}{\lambda_c}, \end{cases} \quad (26)$$

and with standard (essential and/or natural) boundary conditions involving the non-dimensional displacements v^* and u^* , axial interaction $N^* := \frac{N L^2}{\bar{K}_f}$ and bending interaction $M^* := \frac{M L}{\bar{K}_f}$ fields.

6. Case-studies

Parametric analyses are carried out to investigate the electromechanical behavior of piezoelectric nanobeams. The non-dimensional form of the governing structural problem derived in Section 5 is numerically solved for fixed and supported nanobeams. The length is $L = 100$ nm and a rectangular cross section of base $b = 20$ nm and height $h = b$ is assumed. The beam is made of lead zirconate titanate (PZT-5H) with Euler-Young, modulus $E = 126$ GPa, piezoelectric coefficient $e_{zx} = -6.5 \frac{\text{C}}{\text{m}^2}$ and dielectric constant $\lambda_z = 1.3 \cdot 10^{-8} \frac{\text{C}}{\text{Vm}}$ [47].

The effect of the nonlocal parameter λ_c and the electric voltage V on the displacement field is investigated. For fixed piezoelectric nanobeams under a uniformly distributed transverse loading $q^* = 1$, differential equations (25) are solved with prescription of constitutive boundary conditions Eq. (26) and with kinematic boundary conditions on non-dimensional displacements and rotations at beam ends, i.e. $v^* = 0$, $\partial_{\xi} v^* = 0$, $u^* = 0$ at $\xi = \{0, 1\}$. Fig. 1 represents the non-dimensional displacement field for increasing nonlocal parameter, for an applied voltage $V = 0.1$ V. The parametric analyses show a stiffening structural response in agreement with the results provided by [48–50]. The 3-dimensional plot of the displacement field v^* as function of the nonlocal parameter λ_c is provided in Fig. 2.

The maximum non-dimensional displacement $v_{max}^* := v^*(\xi = 0.5)$ versus voltage is represented in Fig. 3 for $\lambda_c = 0.2$, showing an increasing displacement for increasing external voltage. Finally, Fig. 4 shows the 3-dimensional plot of maximum non-dimensional displacement v_{max}^* versus nonlocal parameter λ_c and applied voltage V .

For supported piezoelectric nanobeams under a uniformly distributed transverse loading $q^* = 1$, the differential equations (25) are solved with prescription of constitutive boundary conditions Eq. (26) and with kinematic and static boundary conditions, $v^* = 0$, $M^* =$

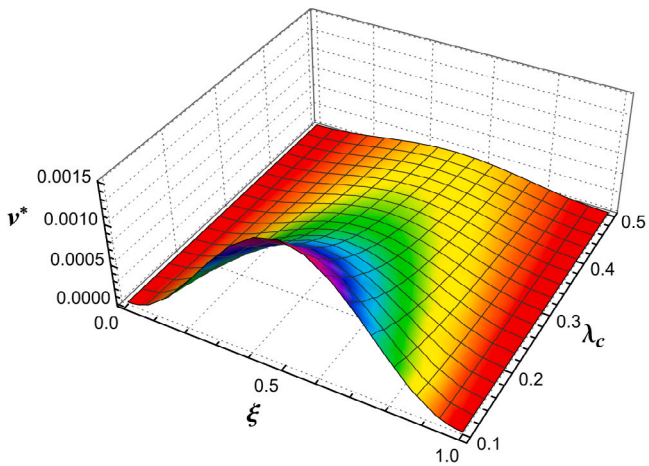


Fig. 2. Fixed piezoelectric nanobeam: 3D plot of non-dimensional displacement field v^* as function of the nonlocal parameter λ_c .

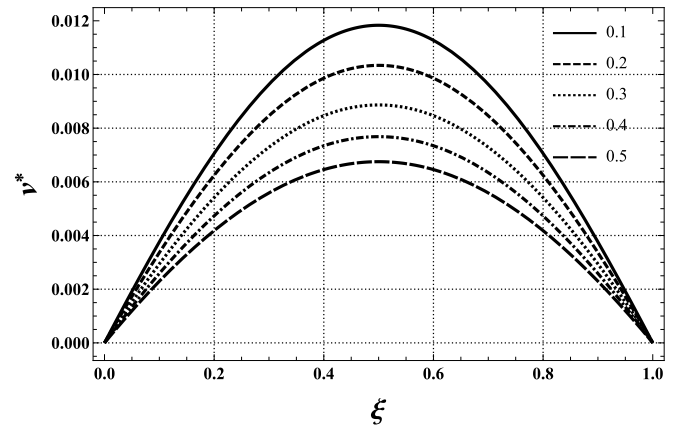


Fig. 5. Supported piezoelectric nanobeam: non-dimensional displacement field v^* for increasing nonlocal parameter λ_c , for an applied voltage $V = 0.1$ V.

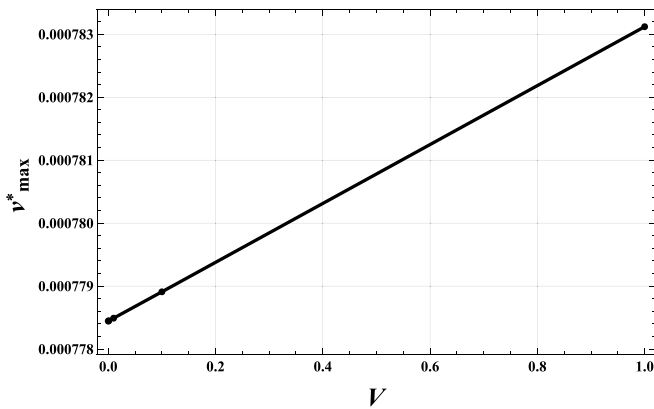


Fig. 3. Fixed piezoelectric nanobeams: maximum non-dimensional displacement $v_{max}^* := v^*(\xi = 0.5)$ versus voltage V for $\lambda_c = 0.2$.

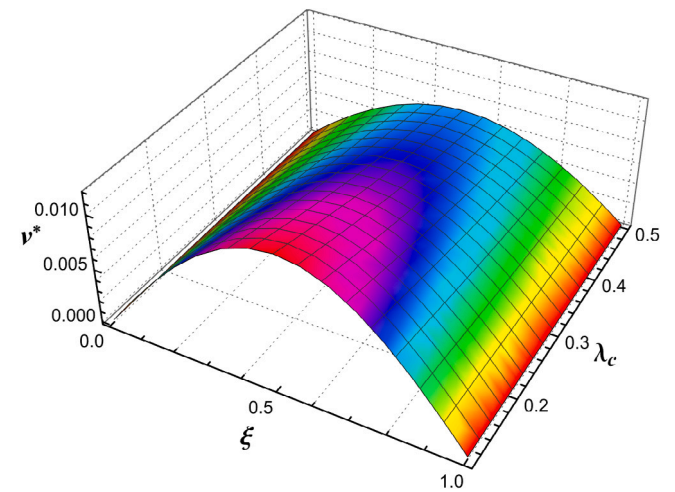


Fig. 6. Supported piezoelectric nanobeam: 3D plot of non-dimensional displacement field v^* as function of nonlocal parameter λ_c .

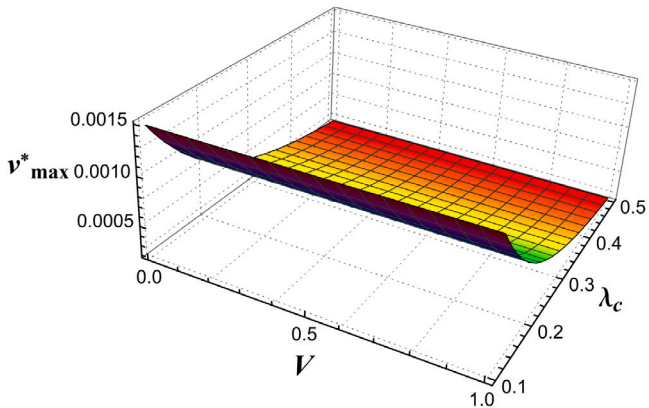


Fig. 4. Fixed piezoelectric nanobeams: 3D plot of maximum non-dimensional displacement v_{max}^* versus nonlocal parameter λ_c and applied voltage V .

$0, u^* = 0$ at $\xi = \{0, 1\}$, with the non-dimensional bending interaction field is given by

$$M^*(\xi) = -\partial_{\xi}^2 v^*(\xi) + \lambda_c^2 \partial_{\xi}^4 v^*(\xi). \tag{27}$$

The non-dimensional displacement field for increasing λ_c , for an applied voltage $V = 0.1$ V is represented in Fig. 5. A stiffening structural behavior is obtained, as shown in the 3-dimensional plot representing

the displacement field v^* as function of the nonlocal parameter λ_c (see Fig. 6).

Fig. 7 shows the maximum non-dimensional displacement $v_{max}^* := v^*(\xi = 0.5)$ versus voltage V , for $\lambda_c = 0.2$. Fig. 8 represents the 3D plot of v_{max}^* versus nonlocal parameter λ_c and applied voltage V , showing a decreasing maximum displacement for increasing nonlocal parameter and for decreasing voltage.

7. Concluding remarks

The electromechanical behavior of piezoelectric nanobeams has been investigated. The Bernoulli–Euler theory has been exploited to formulate kinematics and equilibrium of slender beams, while geometric nonlinearities have been taken into account using the von Kármán formulation. The mechanical and electrostatic coupling has been captured by means of the theory of piezoelectricity. Size effects have been introduced by extending the constitutive piezoelectric law to the nonlocal framework. Notably, the stress-driven integral law has been exploited, providing the nonlocal elastic piezoelectric curvature as spatial convolution driven by the bending interaction field and the effective flexural compliance.

The governing equations for piezoelectric nonlocal nonlinear elastic beams have been established and a non-dimensional form has been provided for computational purposes. The relevant problem has been

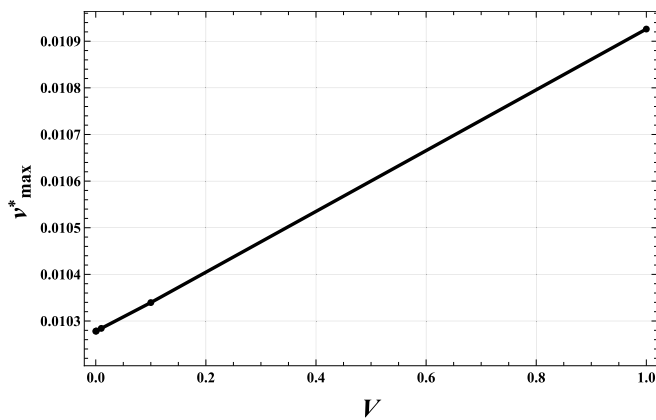


Fig. 7. Supported piezoelectric nanobeams: maximum non-dimensional displacement $v_{max}^* := v^*(\xi = 0.5)$ versus voltage V for $\lambda_c = 0.2$.

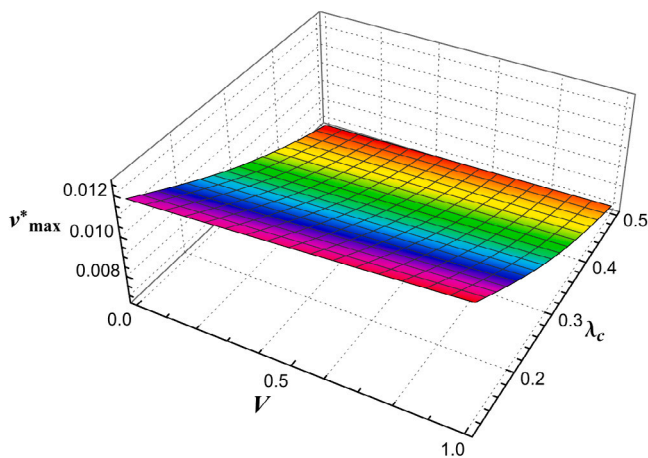


Fig. 8. Supported piezoelectric nanobeams: 3D plot of maximum non-dimensional displacement v_{max}^* versus nonlocal parameter λ_c and applied voltage V .

solved for representative structural schemes. Parametric studies have been performed to assess the effects of nonlocal parameter and applied voltage on the electromechanical response. It has been shown that an increase in the applied electric voltage results in a larger maximum displacement. Conversely, increasing the nonlocal parameter induces a stiffening effect in the structural response, leading to a reduction in the peak displacement. The proposed approach can support and improve the modeling and design of innovative small-scale systems based on piezoelectric soft nanostructures.

CRediT authorship contribution statement

Marzia Sara Vaccaro: Conceptualization, Formal analysis, Investigation, Methodology, Software, Writing – original draft. **Raffaele Barretta:** Conceptualization, Formal analysis, Investigation, Methodology, Supervision, Writing – original draft. **Andrea Caporale:** Formal analysis, Investigation, Software, Writing – original draft. **Raimondo Luciano:** Formal analysis, Investigation, Supervision, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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