First and Second Order Centrodes of Slider-Crank Mechanisms by Using Instantaneous Invariants

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Abstract. A general algorithm to trace the first and second order centrodes of slider-crank mechanisms is proposed by using the instantaneous geometric and kinematic invariants. Bresse's circles can be also traced in order to validate the instantaneous positions of the velocity and acceleration poles. In particular, the second order centrodes give kinematic properties of the coupler motion and, thus, they are computed and traced for a constant angular velocity of the driving crank. Significant examples are included in the paper to validate the proposed algorithm.

Keywords: Instantaneous geometric and kinematic invariants · Slider-crank mechanisms · First and second order centrodes · Coupler motion · Bresse's circles

1 Introduction

Linkages and mainly those derived by four-bar kinematic chains through different kinematic inversions, find several applications in different fields, where they can play the role of kinematic structure of industrial and non-industrial robots, as walking machines, and/or the role of mechanisms to move the fingers of grippers and robotic hands, as described in [1–4]. In particular, the kinematic analysis and synthesis of planar mechanisms can be developed with the aid of suitable geometric loci, as the fixed and moving centrodes and Bresse's circles, as shown in [5–9]. Interesting applications can be also found in spherical and spatial mechanisms, in terms of pitch cones [10] and axodes [11], along with the spherical equivalent of Bresse's circles [12]. Moreover, the instantaneous geometric and kinematic invariants that are directly related to the rigid body motion, can be very useful for the kinematic analysis and synthesis of mechanisms and to determine the main geometric loci of the coupler motion. In fact, these loci take a convenient algebraic form, when they are referred to the canonical frame, as shown in [13–25].

This paper deals with the formulation of a general algorithm to trace the first and second order centrodes of both types of centered and offset slider-crank mechanisms, by using the instantaneous geometric and kinematic invariants. Bresse's circles can be also traced in order to validate the instantaneous positions of the velocity and acceleration

O. Altuzarra and A. Kecskémthy (Eds.): ARK 2022, SPAR 24, pp. 303–310, 2022.
https://doi.org/10.1007/978-3-031-08140-8_33

poles. In particular, the second order centrodes are computed and traced for a constant angular velocity of the driving crank. Significant results are shown in order to validate the proposed algorithm.

2 Instantaneous Invariants

The instantaneous invariants have been introduced in order to evaluate coefficients, constants and parameters that can characterize the motion of a planar mechanisms by geometrical and kinematical points of view.

The instantaneous geometric invariants are related to the rigid motion and the instantaneous kinematic invariants are time-dependent. In particular, the instantaneous geometric invariants are defined invariants among any pairs of fixed and moving reference frames or coordinate systems, since related to the motion characteristics of the coupler link and thus, they are independent by the particular choice of the reference frames.

Referring to the offset slider-crank mechanism of Fig. 1, the pairs of fixed $\mathcal{F}$ (O, X, Y) and moving f (Ω, x, y) reference frames, were chosen along with the corresponding fixed and moving canonical reference frames $\tilde{\mathcal{F}}$ $(P_1, \tilde{X}, \tilde{Y})$ and $\tilde{f}$ $(P_1, \tilde{x}, \tilde{y})$, which origin coincides with the instantaneous center of rotation P_1 of the coupler link AB. The $\tilde{Y}$-axis is orthogonal at P_1 point to the fixed centrode π and oriented toward the moving centrode that is not shown in Fig. 1. Thus, the $\tilde{X}$-axis is tangent to both centrodes at P_1 point and oriented clockwise with respect to the $\tilde{Y}$-axis, while the moving canonical reference frame $\tilde{f}$ is assumed as coincident with $\tilde{\mathcal{F}}$ at the referring configuration. These canonical frames are very important for the kinematic analysis and synthesis of planar mechanisms, because the geometric loci, which are of kinematic interest, take a simple mathematical form, when expressed with respect to them.

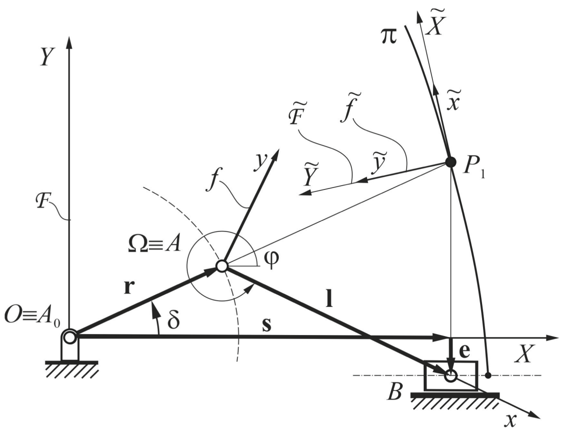


Fig. 1. Original ($\mathcal{F}$ and f) and canonical ($\tilde{\mathcal{F}}$ and $\tilde{f}$) reference frames for a general offset slider-crank mechanism.

The position vector $\mathbf{r}_\Omega$ of the origin Ω of f (Ω, x, y) can be expressed as

$$\mathbf{r}_\Omega = r\,[\cos\delta \;\; \sin\delta\,]^T \tag{1}$$

where r and δ are respectively, the A_0A crank length and the oriented counter-clockwise angle of A_0A with respect to the X-axis. Thus, during the mechanism motion, $\mathcal{F}$ and $\tilde{\mathcal{F}}$ remain fixed to the frame, while f and $\tilde{f}$ move as attached to the coupler link AB of the slider-crank mechanism. The same rigid body motion can be also obtained by rolling the moving on the fixed centrode by giving the successive positions of P_1 as the tangent point between them.

The instantaneous geometric invariants a_n and b_n are the n-order (where n is a natural number) derivatives of the Cartesian-coordinates $\tilde{X}_{P1}$ and $\tilde{Y}_{P1}$ of P_1 with respect to the oriented angle φ of $\tilde{f}$ with respect to $\tilde{\mathcal{F}}$ during the coupler motion

$$a_n = \frac{\mathrm{d}^n \tilde{X}_{P1}}{\mathrm{d}\varphi^n} \quad b_n = \frac{\mathrm{d}^n \tilde{Y}_{P1}}{\mathrm{d}\varphi^n} \tag{2}$$

For the convenient starting configuration of the mechanism, where δ and φ are equal to zero, points P_1 and B are coincident and both canonical frames coincide each other, the instantaneous geometric invariants up to the second order are given by the following expressions

$$a_0 = b_0 = a_1 = b_1 = a_2 = 0 \tag{3}$$

$$b_2 = \sqrt{\left(\frac{\mathrm{d}^2 X_\Omega}{\mathrm{d}\varphi^2} + \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\right)^2 + \left(\frac{\mathrm{d}^2 Y_\Omega}{\mathrm{d}\varphi^2} - \frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\right)^2} \tag{4}$$

Referring to Eq. (1), the first and second derivatives with respect to the oriented angle φ of the Cartesian coordinates X_Ω and Y_Ω that represent the components of the position vector $\mathbf{r}_\Omega$, are given by

$$\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi} = -r\sin\delta\frac{\mathrm{d}\delta}{\mathrm{d}\varphi} \quad \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi} = r\cos\delta\frac{\mathrm{d}\delta}{\mathrm{d}\varphi} \tag{5}$$

$$\frac{\mathrm{d}^2 X_\Omega}{\mathrm{d}\varphi^2} = -r\left[\cos\delta\left(\frac{\mathrm{d}\delta}{\mathrm{d}\varphi}\right)^2 + \sin\delta\frac{\mathrm{d}^2\delta}{\mathrm{d}\varphi^2}\right] \quad \frac{\mathrm{d}^2 Y_\Omega}{\mathrm{d}\varphi^2} = -r\left[\sin\delta\left(\frac{\mathrm{d}\delta}{\mathrm{d}\varphi}\right)^2 - \cos\delta\frac{\mathrm{d}^2\delta}{\mathrm{d}\varphi^2}\right] \tag{6}$$

Similarly, the first and second derivatives of the crank angle δ with respect to φ can be expressed in sequence, as follows

$$\sin\delta = \frac{e - l\sin\varphi}{r}, \quad \frac{\mathrm{d}\delta}{\mathrm{d}\varphi} = -\frac{l\cos\varphi}{r\cos\delta}, \quad \frac{\mathrm{d}^2\delta}{\mathrm{d}\varphi^2} = \frac{l\left(\sin\varphi\cos\delta - \cos\varphi\sin\delta\frac{\mathrm{d}\delta}{\mathrm{d}\varphi}\right)}{r\cos^2\delta} \tag{7}$$

The proposed formulation allows the computations of the instantaneous geometric invariants a_n and b_n for $n = 0, 1, 2$ which are very useful to express in a canonical algebraic form, the most significant geometric loci with respect to $\tilde{f}(P_1, \tilde{x}, \tilde{y})$.

This computation can be very complex when referring to the canonical frames directly, from which the convenience to make use of a different pair of frames, as f and $\mathcal{F}$, which are closer to the mechanism motion than the canonical frames. The first and second derivatives of the crank angle φ with respect to δ can be expressed in sequence, as follows:

$$\varphi = \sin^{-1}\left(\frac{e - r\sin\delta}{l}\right) \quad \dot{\varphi} = \frac{d\varphi}{d\delta} = -\frac{r\,\cos\delta}{l\,\cos\varphi}\dot{\delta} \tag{8}$$

$$\ddot{\varphi} = \frac{d^2\varphi}{d\delta^2} = \frac{\dot{\delta}^2 r\sin\delta\,\cos\varphi - \ddot{\delta} r\,\cos\delta\,\cos\varphi + \frac{\dot{\delta}^2 r^2\cos^2\delta\,\sin\varphi}{l\cos\varphi}}{l\cos^2\varphi} \tag{9}$$

where $\dot{\delta}$ and $\ddot{\delta}$ are the input angular velocity and acceleration of A_0A crank, respectively.

3 First and Second Order Centrodes

The fixed and moving centrodes of first and second order are traced by the instantaneous center of rotation P_1 and the acceleration pole P_2, with respect to the fixed $\mathcal{F}$ and moving f reference frames.

The position of a generic coupler point M can be expressed as

$$X_M = r_{\Omega x} + x_M\cos\varphi - y_M\sin\varphi \quad Y_M = r_{\Omega y} + x_M\sin\varphi + y_M\cos\varphi \tag{10}$$

In particular, when M coincides with P_1, Eqs. (10) give

$$X_{P1} = r_{\Omega x} + x_{P1}\cos\varphi - y_{P1}\sin\varphi \quad Y_{P1} = r_{\Omega y} + x_{P1}\sin\varphi + y_{P1}\cos\varphi \tag{11}$$

and since P_1 is the velocity pole, one has $\frac{dX_{P1}}{dt} = \frac{dY_{P1}}{dt} = 0$, and in turn

$$\begin{aligned}
&\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} - x_{P1}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sin\varphi - y_{P1}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\cos\varphi = 0\\
&\frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} + x_{P1}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\cos\varphi - y_{P1}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sin\varphi = 0
\end{aligned} \tag{12}$$

Thus, Eqs. (12) give

$$x_{P1} = \frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\sin\varphi - \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\cos\varphi \quad y_{P1} = \frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\cos\varphi + \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\sin\varphi \tag{13}$$

and considering the Eqs. (5), one has

$$x_{P1} = -r\frac{\mathrm{d}\delta}{\mathrm{d}\varphi}(\sin\delta\sin\varphi + \cos\delta\cos\varphi) \quad y_{P1} = r\frac{\mathrm{d}\delta}{\mathrm{d}\varphi}(\sin\delta\cos\varphi + \cos\delta\sin\varphi) \tag{14}$$

Substituting the second of Eqs. (7) in Eqs. (14), the following parametric equations of the first order moving centrode l_1 in f are obtained

$$x_{P1} = l\cos\varphi(\tan\delta\sin\varphi + \cos\varphi) \quad y_{P1} = l\cos\varphi(\tan\delta\cos\varphi - \sin\varphi) \tag{15}$$

The parametric equations of the first order fixed centrode λ_1 in $\mathcal{F}$ take the following form

$$X_{P1} = r\cos\delta\left(1 + \frac{l\cos\varphi}{r\cos\delta}\right) \quad Y_{P1} = r\sin\delta\left(1 + \frac{l\cos\varphi}{r\cos\delta}\right) \tag{16}$$

Likewise, when point M coincides with P_2, the second order fixed centrode λ_2 can be expressed as

$$X_{P2} = r_{\Omega x} + x_{P2}\cos\varphi - y_{P2}\sin\varphi \quad Y_{P2} = r_{\Omega y} + x_{P2}\sin\varphi + y_{P2}\cos\varphi \tag{17}$$

and since P_2 is the acceleration pole, one has $\frac{dX_{P2}^2}{dt^2} = \frac{dY_{P2}^2}{dt^2} = 0$, and in turn

$$\begin{aligned}
&\frac{\mathrm{d}^2 X_\Omega}{\mathrm{d}\varphi^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} - x_{P2}\left[\cos\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \sin\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\right] + y_{P2}\left[\sin\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 - \cos\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\right] = 0\\
&\frac{\mathrm{d}^2 Y_\Omega}{\mathrm{d}\varphi^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} - x_{P2}\left[\sin\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 - \cos\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\right] - y_{P2}\left[\cos\varphi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2 + \sin\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}\right] = 0
\end{aligned} \tag{18}$$

Thus, the parametric equations of the second order moving centrode l_2 in f take the following form

$$\begin{aligned}
x_{P2} =& \frac{1}{\dot\varphi^4 + \ddot\varphi^2}\Bigg[\left(\frac{\mathrm{d}^2X_\Omega}{\mathrm{d}\varphi^2}\cos\varphi + \frac{\mathrm{d}^2Y_\Omega}{\mathrm{d}\varphi^2}\sin\varphi\right)\dot\varphi^4 + \Bigg(\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\cos\varphi + \frac{\mathrm{d}^2X_\Omega}{\mathrm{d}\varphi^2}\sin\varphi\\
&+ \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\sin\varphi - \frac{\mathrm{d}^2Y_\Omega}{\mathrm{d}\varphi^2}\cos\varphi\Bigg)\dot\varphi^2\ddot\varphi + \left(\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\sin\varphi - \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\cos\varphi\right)\ddot\varphi^2\Bigg]
\end{aligned} \tag{19}$$

$$\begin{aligned}
y_{P2} =& \frac{1}{\dot\varphi^4 + \ddot\varphi^2}\Bigg[\left(-\frac{\mathrm{d}^2X_\Omega}{\mathrm{d}\varphi^2}\sin\varphi + \frac{\mathrm{d}^2Y_\Omega}{\mathrm{d}\varphi^2}\cos\varphi\right)\dot\varphi^4 + \Bigg(-\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\sin\varphi + \frac{\mathrm{d}^2X_\Omega}{\mathrm{d}\varphi^2}\cos\varphi\\
&+ \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\cos\varphi + \frac{\mathrm{d}^2Y_\Omega}{\mathrm{d}\varphi^2}\sin\varphi\Bigg)\dot\varphi^2\ddot\varphi + \left(\frac{\mathrm{d}X_\Omega}{\mathrm{d}\varphi}\cos\varphi + \frac{\mathrm{d}Y_\Omega}{\mathrm{d}\varphi}\sin\varphi\right)\ddot\varphi^2\Bigg]
\end{aligned} \tag{20}$$

It is well known, that the fixed and moving centrodes of first order are always tangent each other at the instantaneous center of rotation and that the rigid body motion can be reproduced by the pure rolling of the moving centrode on the fixed one. Instead, the fixed and moving centrodes of second order intersect each other in more than one point, for which the position of the acceleration pole is not straightforward to determine. However, one way to find the acceleration pole is to use Bresse's circles because they intersect each other in both poles P_1 and P_2.

Thus, the inflection and stationary circles are determined by still using the instantaneous invariants and even for validation purposes, because the acceleration pole must be one of the intersection points between the second order centrodes, but also one of the two intersection between Bresse's circles.

In particular, the inflection circle $\mathcal{I}$ is the geometric locus of the coupler points, which show an inflection point in their paths and is always tangent to both centrodes at the instantaneous center of rotation P_1. Referring to $\tilde{f}$, one has

$$\tilde{x}^2 + \tilde{y}^2 - b_2\,\tilde{y} = 0 \tag{21}$$

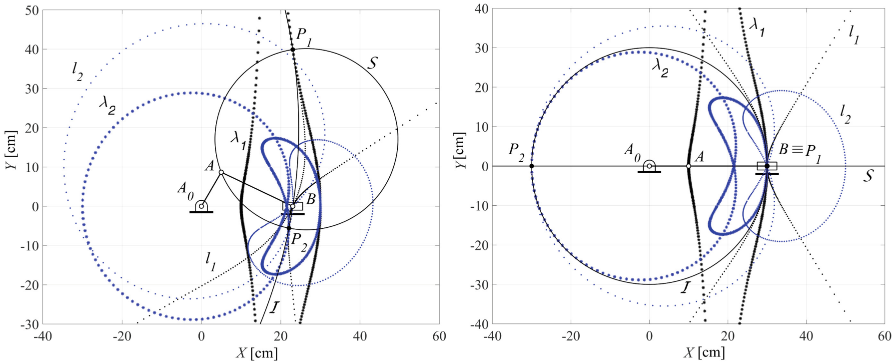
where the diameter b_2 of $\mathcal{I}$ is obtained by the Eq. (4).

Likewise, the stationary circle $\mathcal{S}$ that is the geometric locus of the coupler points, which show a pure normal acceleration, is given by

$$\ddot\varphi\left(\tilde{x}^2 + \tilde{y}^2\right) + b_2\,\dot\varphi^2\,\tilde{x} = 0 \tag{22}$$

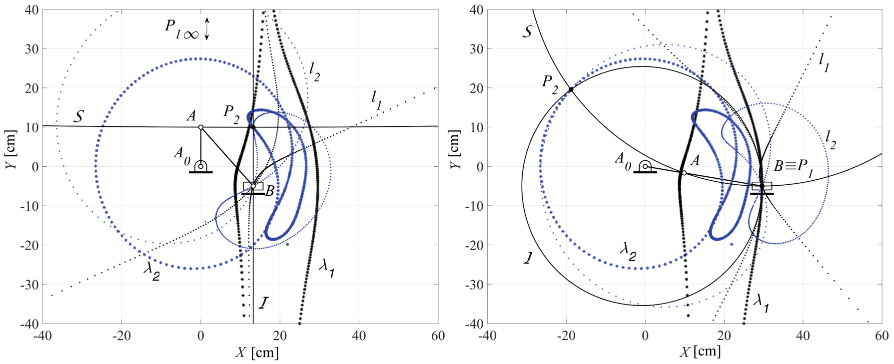
where $\dot{\varphi}$ and $\ddot{\varphi}$ are the angular velocity and acceleration, respectively, which are expressed by Eqs. (8) and (9). Equations (19), (20) and (22) depend by the kinematic properties by means of $\dot{\varphi}$ and $\ddot{\varphi}$, while the inflection circle is related to the geometry of the coupler motion.

The proposed algorithm has been implemented in Matlab and validated through several significant examples dealing with centered and offset slider-crank mechanisms. In particular, Figs. 2 and 3 have been obtained for $r = 10$ cm, $l = 2r$, $\dot{\delta} = 1$ r/s, $\ddot{\delta} = 0$ and e (offset) $= -50$ cm. When $\delta = 0$ deg, point B is at top dead center and thus, it coincides with P_1, as shown in Fig. 2b) and for $\delta = 90$ deg, both Bresse's circles degenerates in two orthogonal straight lines, Fig. 3a).



a) b)

Fig. 2. Fixed and moving centrodes of first and second order, along with the inflection and stationary circles, for a centered slider-crank mechanism: a) $\delta = 60$ deg; b) $\delta = 0$ deg.



a) b)

Fig. 3. Fixed and moving centrodes of first and second order, along with the inflection and stationary circles of an offset slider-crank mechanism: a) $\delta = 90$ deg; b) Top dead center.

4 Conclusions

A general algorithm to trace the first and second order centrodes of centered and offset slider-crank mechanisms, along with Bresse's circles, has been formulated by using the instantaneous geometric and kinematic invariants. In particular, the second order centrodes have been computed and traced for a constant angular velocity of the driving crank, otherwise the fixed and moving second order centrodes would change accordingly, by losing their practical meaning. The proposed algorithm was implemented in a Matlab program and significant examples have been obtained and discussed in order to validate the proposed algorithm.

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