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Hilbert series of simple thin polyominoes

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Abstract
Let $\mathcal{P}$ be a simple thin polyomino, namely a polyomino that has no holes and does not contain a square tetromino as a subpolyomino. In this paper, we determine the reduced Hilbert–Poincaré series $h(t)/(1-t)^d$ of $K[\mathcal{P}]$ by proving that $h(t)$ is the rook polynomial of $\mathcal{P}$. As an application, we characterize the Gorenstein simple thin polyominoes.

Keywords Simple polyominoes · Hilbert–Poincaré series · Rook polynomial · Gorenstein algebras

Mathematics Subject Classification 13D40 · 05B50

1 Introduction

Polyominoes are two-dimensional objects obtained by joining edge by edge squares of same size, and they are studied from the point of view of combinatorics, e.g., in tiling problems of the plane (see [6]). Recently, in [13], Qureshi introduced a binomial ideal induced by the geometry of a given polyomino $\mathcal{P}$, called polyomino ideal, and the related algebra $K[\mathcal{P}]$ (see Sect. 2). From that moment, different authors studied algebraic properties related to this ideal (see [9,11,14,16]). In particular, in [9,14] the authors proved that if $\mathcal{P}$ is simple, namely the polyomino has no holes, then $K[\mathcal{P}]$ is a Cohen–Macaulay domain.

In this paper, we compare two generating functions associated with polyominoes: the Hilbert series of $K[\mathcal{P}]$ and the rook polynomial of $\mathcal{P}$ (see [15, Chapter 7]). The well-known "rook problem" is the problem of enumerating the number of ways of placing k non-attacking rooks on a chessboard. In a similar way, let $\mathcal{P}$ be a polyomino

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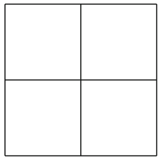


Fig. 1 The square tetromino

and let r_k be the number of ways of arranging k non-attacking rooks on the cells of $\mathcal{P}$. The polynomial

$$r_{\mathcal{P}}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k$$

is called the *rook polynomial* of $\mathcal{P}$ and $r(\mathcal{P})$ is called the *rook number* of $\mathcal{P}$.

In a recent paper [4], the authors proved that, for particular convex polyominoes $\mathcal{P}$, the Castelnuovo–Mumford regularity of $K[\mathcal{P}]$ is equal to $r(\mathcal{P})$. Starting from this result, we consider the Hilbert–Poincaré series of simple polyominoes as a nice object to grasp the above equality and other fundamental invariants by using elementary proofs.

We say that a polyomino $\mathcal{P}$ is *thin* (see [12]) if $\mathcal{P}$ does not contain the square tetromino (see Fig. 1) as a subpolyomino.

One of the main results of this paper is the following

Theorem 1.1 *Let $\mathcal{P}$ be a simple thin polyomino such that the reduced Hilbert–Poincaré series of $K[\mathcal{P}]$ is*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{h(t)}{(1-t)^d}.$$

Then, $h(t)$ is the rook polynomial of $\mathcal{P}$.

In particular, it follows that the Castelnuovo–Mumford regularity of $K[\mathcal{P}]$ is $r(\mathcal{P})$ and the multiplicity of $K[\mathcal{P}]$ is $r_{\mathcal{P}}(1)$. Theorem 1.1 gives us information on the Hilbert series and the Castelnuovo–Mumford regularity of the toric ring related to the bipartite graph $G_{\mathcal{P}}$ induced in a natural way by a simple polyomino $\mathcal{P}$ (see [13, Section 2] and Sect. 2). The condition that $\mathcal{P}$ is thin translates to the condition that the bipartite graph $G_{\mathcal{P}}$ does not contain $K_{3,3}$ as a subgraph, where $K_{3,3}$ is the complete bipartite graph with two parts of equal size 3.

An open question is to give a complete characterization of the Gorensteinness of the algebra $K[\mathcal{P}]$ when $\mathcal{P}$ is a simple polyomino. Some partial results in this direction are in [1,4,13]. The other main result of this paper is Theorem 4.2, in which we classify the simple thin polyominoes $\mathcal{P}$ having a Gorenstein algebra $K[\mathcal{P}]$, due to the geometric properties of $\mathcal{P}$. At the end, we present a conjecture and an open question.

2 Preliminaries

In this section, we recall general definitions and notation on polyominoes and algebraic invariants of commutative algebra (see also [7,18]).

Let $a = (i, j), b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$, the set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an *interval* of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$. In particular, a, b are called *diagonal corners* and c, d *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the left lower (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a + (1, 0), a + (0, 1), a + (1, 1)$, and its edges $E(C)$ are

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}.$$

In the following, we denote by $e(C)$ the left lower corner of a cell C.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then, C and D are said to be *connected*, if there is a sequence of cells $C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m-1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \ldots, C_m$ is called a *path* (connecting C and D). A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P}) = \cup_{C \in \mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and we denote it by $\operatorname{rk} \mathcal{P}$. A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$. We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \ldots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \ldots, m$. An interval $[a, b]$ with $a = (i, j)$ and $b = (k, \ell)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = \ell$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \ldots, k - 1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it *maximal horizontal edge interval*. Similarly, one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$.

Let $\mathcal{P}$ be a polyomino and define the polynomial ring $R = K[x_v \mid v \in V(\mathcal{P})]$ over a field K. The binomial $x_a x_b - x_c x_d \in R$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where c, d are the anti-diagonal corners of $[a, b]$. We denote by $\mathcal{M}$ the set of all inner 2-minors of $\mathcal{P}$. The ideal $I_{\mathcal{P}} \subset R$ generated by $\mathcal{M}$ is called the *polyomino ideal* of $\mathcal{P}$. We also set $K[\mathcal{P}] = R/I_{\mathcal{P}}$.

By combining [9, Theorem 2.1] with [8, Corollary 3.3], one obtains the following

Lemma 2.1 *Let $\mathcal{P}$ be a simple polyomino. Then, $K[\mathcal{P}]$ is a normal Cohen–Macaulay domain of Krull dimension $|V(\mathcal{P})| - \operatorname{rk} \mathcal{P}$.*

We recall that given a simple polyomino $\mathcal{P}$, the K-algebra $K[\mathcal{P}]$ coincides with the toric ring of a bipartite graph $G_{\mathcal{P}}$ defined as follows. Let $\{V_1, \ldots, V_m\}$ be the set of the maximal vertical edge intervals of $\mathcal{P}$ and $\{H_1, \ldots, H_n\}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$ (see Fig. 2). We denote by $G_{\mathcal{P}}$ the associated bipartite graph of $\mathcal{P}$ with vertex set $\{v_1, \dots, v_m\} \cup \{h_1, \dots, h_n\}$ and edge set

$$E(G_{\mathcal{P}}) = \{\{v_i, h_j\} : V_i \cap H_j \in V(\mathcal{P})\}.$$

The graph $G_{\mathcal{P}}$ is known to be weakly chordal (see [14, Lemma 2.1]). For further information on toric ideal of graphs, see also [2,10]. Let $T = K[v_i h_j : \{v_i, h_j\} \in E(G_{\mathcal{P}})] \subset K[v_1, \dots, v_m, h_1, \dots, h_n]$ be the toric ring of $G_{\mathcal{P}}$. We denote by a_{ij} the vertex of $\mathcal{P}$ that lies on the intersection of the edge intervals V_i and H_j, and we denote by x_{ij} the variable of R associated with a_{ij}. Let $\varphi : R \to T$ be the K-algebra homomorphism defined by $\varphi(x_{ij}) = v_i h_j$, we set $J_{\mathcal{P}} = \ker \varphi$. According to Lemma 2.1, $J_{\mathcal{P}} = I_{\mathcal{P}}$, that is $K[\mathcal{P}] \cong T$. Thanks to the above interpretation and [2, Theorem 4.9], we obtain an upper bound for the Castelnuovo–Mumford regularity of $K[\mathcal{P}]$.

Lemma 2.2 *Let $\mathcal{P}$ be a simple polyomino, and let $\{V_1, \dots, V_m\}$ be the set of the maximal vertical edge intervals of $\mathcal{P}$ and $\{H_1, \dots, H_n\}$ be the set of the maximal horizontal edge intervals of $\mathcal{P}$. Then,*

$$\operatorname{reg} K[\mathcal{P}] \leq \min\{m, n\} - 1.$$

The bound in Lemma 2.2 could be far from the value of reg $K[\mathcal{P}]$, as one can see in the polyomino in Fig. 2.

Let R be a standard graded ring and I be a homogeneous ideal. The *Hilbert function* $\mathrm{H}_{R/I} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathrm{H}_{R/I}(k) := \dim_K (R/I)_k$$

where $(R/I)_k$ is the k-degree component of the gradation of R/I, while the *Hilbert–Poincaré series* of R/I is

$$\mathrm{HP}_{R/I}(t) := \sum_{k \in \mathbb{N}} \mathrm{H}_{R/I}(k) t^k.$$

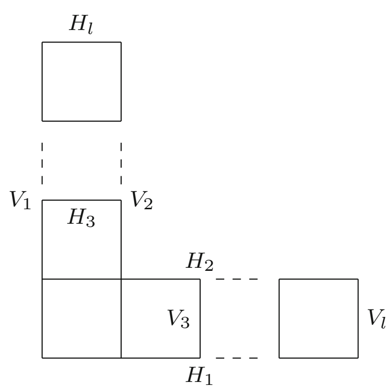


Fig. 2 A polyomino $\mathcal{P}$ with reg $K[\mathcal{P}] = 2$ and $m = n = l >> 2$

By the Hilbert–Serre theorem, the Hilbert–Poincaré series of R/I is a rational function. In particular, by reducing this rational function we get

$$\mathrm{HP}_{R/I}(t) = \frac{h(t)}{(1-t)^d}.$$

for some $h(t) \in \mathbb{Z}[t]$, where d is the Krull dimension of R/I. The degree of $\mathrm{HP}_{R/I}(t)$ as a rational function, namely $\deg h(t) - d$, is called *a-invariant* of R/I, denoted by $a(R/I)$. It is known that whenever R/I is Cohen–Macaulay, we have $a(R/I) = \operatorname{reg} R/I - \operatorname{depth} R/I$, that is $\operatorname{reg} R/I = \deg h(t)$.

We recall the following result about Hilbert series

Proposition 2.3 *Let I be a homogeneous ideal of a graded ring R, let $f \in R$ be a homogeneous element of degree d and consider the following exact sequence.*

$$0 \longrightarrow R/(I:f) \xrightarrow{\cdot f} R/I \longrightarrow R/(I,f) \longrightarrow 0$$

Then,

1. $\mathrm{HP}_{R/I}(t) = \mathrm{HP}_{R/(I,f)}(t) + t^d \mathrm{HP}_{R/(I:f)}(t)$
2. *If f is a regular element, then*

$$\mathrm{HP}_{R/I}(t) = \frac{1}{1-t^d}\mathrm{HP}_{R/(I,f)}(t).$$

We also rephrase the result of Stanley [17, Theorem 4.4] that is fundamental for our aim in Sect. 4.

Theorem 2.4 *Let $R = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring, I be a homogeneous ideal of R such that R/I is a Cohen–Macaulay domain, and let*

$$\mathrm{HP}_{R/I}(t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^d}$$

be the reduced Hilbert series of R/I. Then, R/I is Gorenstein if and only if for any $i = 0, \ldots, s$, we have $h_i = h_{s-i}$.

3 Hilbert series of simple thin polyominoes

In this section, we compute the Hilbert series of simple thin polyominoes in relation to their rook polynomial. We start with the following

Definition 3.1 Let C and D be two cells of $\mathbb{N}^2$ such that $e(C) \leq e(D)$. We call the set

$$[C, D] = \{F \in \mathbb{N}^2 : e(F) \in [e(C), e(D)]\}$$

interval of cells. If $e(C)$ and $e(D)$ lie either on the same vertical edge interval or on the same horizontal edge interval, we call $[C, D]$ a *cell interval*. We call $[C, D]$ *inner interval of cells* of $\mathcal{P}$ if any cell in $[C, D]$ is a cell of $\mathcal{P}$.

Lemma 3.2 *Let $\mathcal{P}$ be a simple thin polyomino. Then, any maximal inner interval I of cells of $\mathcal{P}$ is a cell interval, and for any maximal inner interval $J \neq I$ such that $V(I) \cap V(J) \neq \emptyset$, I and J have either one cell, one edge or one vertex in common.*

Proof Since $\mathcal{P}$ does not contain a square tetromino, then also any maximal inner interval of $\mathcal{P}$ does not contain a square tetromino; namely, it is a cell interval.

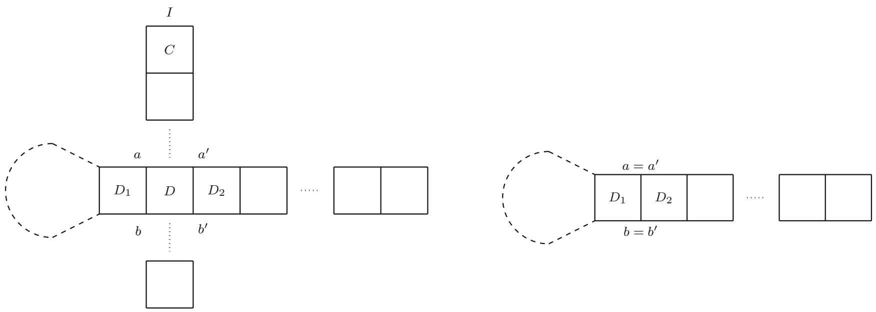
Let I, J be two maximal inner intervals of $\mathcal{P}$ such that $V(I) \cap V(J) \neq \emptyset$. By contradiction, we consider the following two cases: I and J have two or more edges in common, not belonging to the same cell, and I and J have two or more cells in common. In the first case, without loss of generality $V(I) \cap V(J) = [(i, j), (k, j)]$ with $k > i + 1$. Therefore, the cells whose left lower corners are $(i, j-1)$, $(i+1, j-1)$, (i, j), $(i+1, j)$ form a square tetromino, that is a contradiction. In the second case, $I \cup J$ is a maximal inner interval strictly containing I and J, and this is a contradiction. The assertion follows. □

From now on, we will briefly call inner intervals the inner intervals of cells of a polyomino $\mathcal{P}$. In the following, we define the simple polyominoes $\mathcal{P}'$ and $\mathcal{P}''$ obtainable from a simple (thin) polyomino $\mathcal{P}$. The latter are fundamental for the computation of the Hilbert series.

Definition 3.3 (Polyomino $\mathcal{P}'$) Let $\mathcal{P}$ be a simple polyomino. We say that a cell C of $\mathcal{P}$ is a *leaf* if there exists an edge $\{u, v\}$ of C such that $\{u, v\} \cap V(\mathcal{P} \setminus \{C\}) = \emptyset$. We call the vertices u and v *leaf corners* of C. We define the polyomino $\mathcal{P}'$ as the polyomino $\mathcal{P} \setminus \{C\}$.

Definition 3.4 (Polyomino $\mathcal{P}''$) Let $\mathcal{P}$ be a simple thin polyomino, and let I be a maximal inner interval of $\mathcal{P}$. We say that $\mathcal{P}$ is *collapsible* in I if there exists one and only one maximal inner interval J of $\mathcal{P}$ intersecting I in a cell, and $\mathcal{P} = \mathcal{P}_1 \sqcup I \sqcup \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two polyominoes such that $\mathcal{P}_2$ is either empty or a cell interval. When $\mathcal{P}_2$ is empty, I is called a *tail*. When $\mathcal{P}_2$ is a cell interval, I is called an *endcut*. We define the polyomino $\mathcal{P}''$ as follows. Let D be the cell such that $I \cap J = \{D\}$, and let $\{a, b, a', b'\}$ be the corners of D where $a, b \in V(\mathcal{P}_1)$ and $a', b' \in V(\mathcal{P}_2)$. We define $\mathcal{P}''$ as the polyomino obtained from $\mathcal{P} \setminus I$ by the identification of the vertices a and b of $\mathcal{P}_1$ with the vertices a' and b' of $\mathcal{P}_2$, respectively, due to the translation of the cell interval $\mathcal{P}_2$ (see Fig. 3).

Remark 3.5 Let $\mathcal{P}$ be a simple thin polyomino collapsible in I with leaf C. We observe that $r(\mathcal{P}') \in \{r(\mathcal{P}), r(\mathcal{P}) - 1\}$ and $r(\mathcal{P}'') = r(\mathcal{P}) - 1$. For example, if $\mathcal{P}$ is the polyomino in Fig. 6 and we consider the leaf C_1, then $r(\mathcal{P}')$ is equal to $r(\mathcal{P}) - 1$. On the other hand, if $\mathcal{P}$ is the polyomino in Fig. 4 and we consider the leaf containing u and v, then $r(\mathcal{P}')$ is equal to $r(\mathcal{P})$. In both cases, we have $r(\mathcal{P}'') = r(\mathcal{P}) - 1$. In general, if C belongs to any set of $r(\mathcal{P})$ non-attacking rooks, then any set of non-attacking rooks of maximal cardinality in $\mathcal{P}'$ has $r(\mathcal{P}) - 1$ elements. Otherwise, there exists some set of non-attacking rooks of maximal cardinality in $\mathcal{P}'$ having $r(\mathcal{P})$ elements. Moreover, any set of $r(\mathcal{P})$ non-attacking rooks has an element on I, that is $r(\mathcal{P}'') = r(\mathcal{P}) - 1$.



(A) A simple thin polyomino $\mathcal{P}$ which is collapsible in the endcut I

(B) The polyomino $\mathcal{P}''$ after the collapsing of $\mathcal{P}$ on I

Fig. 3 The collapsing operation on a simple thin polyomino $\mathcal{P}$

We now want to prove that any simple thin polyomino is collapsible in some inner interval I. For this aim, we first prove the following

Lemma 3.6 *Let $\mathcal{P}$ be a simple thin polyomino that is not a cell interval. Then, there exists a maximal inner interval I of $\mathcal{P}$ for which there exists one and only one maximal inner interval J of $\mathcal{P}$ intersecting I in a cell.*

Proof Since $\mathcal{P}$ is simple and thin, we observe that for any two cells C and D of $\mathcal{P}$, there is a unique path of cells connecting C and D. By contradiction, assume that for any maximal inner interval of $\mathcal{P}$, there are at least two maximal inner intervals intersecting it in one cell. We show that there exist two different paths connecting two given cells. For this aim, let I be a maximal inner interval of $\mathcal{P}$. There exist I_1 and J such that $I_1 \cap I$ and $I_1 \cap J$ are cells of $\mathcal{P}$. Furthermore, there exists $I_2 \neq I$ intersecting I_1 in one cell. By using the same argument, we find a sequence of inner intervals $I_1, I_2, \ldots$ of $\mathcal{P}$ such that I_j and I_{j+1} have a cell in common. Since the number of inner intervals of $\mathcal{P}$ is finite, then there exists k such that $I_k = J$, and hence, there are two paths connecting a cell C of $I \setminus I \cap J$ with a cell D of $J \setminus I \cap J$, one passing through $I_1, \ldots, I_{k-1}$ and one passing through the cell $I \cap J$. This is a contradiction, and the assertion follows. □

Proposition 3.7 *Let $\mathcal{P}$ be a simple thin polyomino that is not a cell interval. Then, $\mathcal{P}$ is collapsible in some maximal inner interval I.*

Proof If $\mathcal{P}$ has a tail, then the assertion follows. Therefore, assume that $\mathcal{P}$ does not contain tails. By contradiction, assume that $\mathcal{P}$ has no endcuts. From Lemma 3.6, there exists a maximal inner interval I_1 of $\mathcal{P}$ for which there exists one and only one inner interval J_1 of $\mathcal{P}$ intersecting I_1 in one cell. Let $\mathcal{P} = \mathcal{P}_1 \sqcup I_1 \sqcup \mathcal{P}_2$. Since I_1 is not an endcut, then $\mathcal{P}_2$ is a simple thin polyomino that is not a cell interval. Moreover, $\operatorname{rk} \mathcal{P}_2 < \operatorname{rk} \mathcal{P}$. Again from Lemma 3.6, there exists an inner interval I_2 in $\mathcal{P}_2$ for which there exists one and only one inner interval J_2 of $\mathcal{P}$ intersecting I_2 in one cell. We write $\mathcal{P} = \mathcal{P}_3 \sqcup I_2 \sqcup \mathcal{P}_4$, with $\mathcal{P}_1 \subset \mathcal{P}_3$. We repeat the same argument for the simple thin polyomino $\mathcal{P}_4$ with $\operatorname{rk} \mathcal{P}_4 < \operatorname{rk} \mathcal{P}_2$. By proceeding in this way, since the $\operatorname{rk} \mathcal{P}$ is finite, at the end we find an inner interval I_k for which $\mathcal{P} = \mathcal{P}_{2k-1} \sqcup I_k \sqcup \mathcal{P}_{2k}$ such that $\operatorname{rk} \mathcal{P}_{2k} = 0$, namely I_k is a tail, that is a contradiction. □

We observe that the interval I in Lemma 3.6 in which $\mathcal{P}$ is collapsible has one leaf C.

Lemma 3.8 *Let $\mathcal{P}$ be a simple polyomino with a leaf C having leaf corners u and v, and let $\mathcal{P}'$ be as in Definition 3.3. Then, $((I_{\mathcal{P}}, x_u) : x_v) = I_{\mathcal{P}'} + J$ where J is a monomial ideal generated in degree one.*

Proof Since C is a leaf of $\mathcal{P}$, then there exists a maximal cell interval I of $\mathcal{P}$ such that $C \in I$. Let $E = \{u_1, u_2, \dots, u_r, u\}$ and $F = \{v_1, \dots, v_r, v\}$ be the edge intervals of length $r + 1$ of I. We observe that the ideal $I_{\mathcal{P}}$ is generated by the inner 2-minors of $\mathcal{P}' = \mathcal{P} \setminus \{C\}$ and by the inner 2-minors of I whose inner intervals contain the cell C, namely

$$I_{\mathcal{P}} = I_{\mathcal{P}'} + (\{x_v x_{u_i} - x_u x_{v_i}\}_{i=1,\dots,r}).$$

Then,

$$(I_{\mathcal{P}}, x_u) = I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\dots,r}) + (x_u).$$

The thesis follows if we prove that $(I_{\mathcal{P}}, x_u) : x_v \subseteq I_{\mathcal{P}'} + (x_{u_1}, \dots, x_{u_r}, x_u)$, since the other inclusion is trivial. If $f \in (I_{\mathcal{P}}, x_u) : x_v$, then $x_v f \in I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\dots,r}) + (x_u)$, that is

$$x_v f = g + x_v g' + x_u g''$$

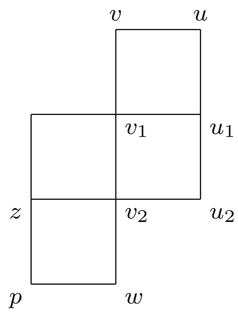
where $g \in I_{\mathcal{P}'}$, $g' \in (x_{u_1}, \dots, x_{u_r})$ and, $g'' \in R$. That is, $x_v(f - g') \in I_{\mathcal{P}'} + (x_u)$ and $f - g' \in (I_{\mathcal{P}'} + (x_u)) : x_v$. Since $\mathcal{P}'$ is simple, then $I_{\mathcal{P}'}$ is prime, and since x_u is not a variable of $I_{\mathcal{P}'}$, then also $I_{\mathcal{P}'} + (x_u)$ is prime. Therefore, since $x_v \notin I_{\mathcal{P}'} + (x_u)$, then $f - g' \in I_{\mathcal{P}'} + (x_u)$ and the assertion follows. □

Remark 3.9 By using the notation of Lemma 3.8, we want to remark that the ideal in the statement has different behaviors, depending on the choice of u and v. Let $\mathcal{P}$ be the simple thin polyomino in Fig. 4, namely the skew tetromino.

Since $x_v x_{u_2} - x_u x_{v_2} \in I_{\mathcal{P}}$, then $x_u x_{v_2} \in (I_{\mathcal{P}}, x_v)$ and $x_{v_2} \in (I_{\mathcal{P}}, x_v) : x_u$. Therefore, since $x_p x_{v_2} - x_w x_z \in I_{\mathcal{P}}$, then $x_w x_z \in (I_{\mathcal{P}}, x_v) : x_u$; namely, $(I_{\mathcal{P}}, x_v) : x_u$ has a monomial generator of degree 2. Nevertheless, the ideal $(I_{\mathcal{P}}, x_u) : x_v$ has no monomial generators of degree greater than 1.

Lemma 3.10 *Let $\mathcal{P}$ be a simple thin polyomino, collapsible in I that has r cells, and let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}', \mathcal{P}''$ be as in Definitions 3.3 and 3.4. Let C be a leaf of I with leaf corners u and v, and assume that $E = \{u_1, u_2, \dots, u_r, u\}$ is the edge interval of I such that $E \cap V(\mathcal{P}_1) = \varnothing$. Then, $R/(I_{\mathcal{P}}, x_u, x_v) \cong K[\mathcal{P}']$ and $R/((I_{\mathcal{P}}, x_u) : x_v) \cong K[\mathcal{P}''] \otimes K[y_1, \dots, y_{r-1}]$.*

Fig. 4 The skew tetromino



Proof Let $F = \{v_1, \dots, v_r, v\}$ be the other edge interval of I of length $r + 1$. By the proof of Lemma 3.8, we have

$$I_{\mathcal{P}} = I_{\mathcal{P}'} + (\{x_v x_{u_i} - x_u x_{v_i}\}_{i=1,\dots,r}).$$

and

$$(I_{\mathcal{P}}, x_u) = I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\dots,r}, x_u).$$

Since $\{u, v\} \cap V(\mathcal{P}') = \varnothing$, then $(I_{\mathcal{P}}, x_u, x_v) = (I_{\mathcal{P}'}, x_u, x_v)$, that is $R/(I_{\mathcal{P}}, x_u, x_v) \cong K[\mathcal{P}']$.

Now, let $I'' = ((I_{\mathcal{P}}, x_u) : x_v)$. By the proof of Lemma 3.8, it arises $I'' = I_{\mathcal{P}'} + (x_{u_1}, \dots, x_{u_r}, x_u)$. Let us consider J and D as in Definition 3.4, with $V(D) = \{u_k, u_{k+1}, v_k, v_{k+1}\}$. We can split J into the cell intervals J_1 and J_2, such that $J_1 \subseteq \mathcal{P}_1$, $\mathcal{P}_2 = J_2$, and the cell D. Since the variables $x_{u_1}, \dots, x_{u_r}, x_u$ are generators of I'', then all of the inner 2-minors of the interval I, and all of the inner 2-minors of J having corners on u_k, u_{k+1}, are redundant. Since $\mathcal{P}_2$ is either empty or a cell interval, then the edge E is a maximal edge interval of $\mathcal{P}$ (see also Remark 3.9). We want to prove that I'' has no minimal monomial generators of degree greater than 1. By Lemma 3.8, assume that there exists a minimal generator $x_w x_z \in I''$, with $w, z \notin \{u_1, \dots, u_r, u\} = E$. That is, there exists $i \in \{1, \dots, r\}$ and $p \in V(\mathcal{P})$ such that $g = x_w x_z - x_{u_i} x_p$ is an inner 2-minor of $\mathcal{P}$. That is, one between w and z, say w, lies on the same edge interval containing the u_i's and $w \notin E$; namely, $E \cup \{w\}$ is an edge interval of $\mathcal{P}$ containing E; that is, E is not a maximal, contradiction.

If $\mathcal{P}_2$ is empty, from Definition 3.4 we have $\mathcal{P}'' = \mathcal{P} \setminus I = \mathcal{P}_1$. Since $E \cap V(\mathcal{P}_1) = \varnothing$, then $I'' = I_{\mathcal{P}_1} + (x_{u_1}, \dots, x_{u_r}, x_u)$, $V(\mathcal{P}'') \cap F = \{v_k, v_{k+1}\}$, and therefore,

$$R/I'' \cong K[\mathcal{P}''] \otimes K[x_{v_1}, \dots, x_{v_{k-1}}, x_{v_{k+2}} \dots, x_{v_r}, x_v]$$

and the assertion follows. Otherwise, let $\mathcal{P}''$ be the polyomino arising from the translation of the edge $\{u_k, u_{k+1}\}$ on the edge $\{v_k, v_{k+1}\}$. We want to prove that $I'' = I_{\mathcal{P}''} + (x_{u_1}, \dots, x_{u_r}, x_u)$.

Let $f = f^+ - f^- \in I''$ be an irreducible binomial, and let

$$V(f) = \{v \in V(\mathcal{P}) \mid x_v | f^+ \text{ or } x_v | f^-\}.$$

One of the following is true

(a) $V(f) \subseteq V(\mathcal{P}_1)$ or $V(f) \subseteq V(\mathcal{P}_2) \setminus \{u_k, u_{k+1}\}$;
(b) $|V(f) \cap V(\mathcal{P}_1)| = |V(f) \cap V(\mathcal{P}_2) \setminus \{u_k, u_{k+1}\}| = 2$.

In case (a), we have $f \in I_{\mathcal{P}''}$.
In case (b), since J is the unique maximal cell interval having non-empty intersection with both $\mathcal{P}_1$ and $\mathcal{P}_2$, we have that $|V(f) \cap V(J_1)| = |V(f) \cap V(J_2) \setminus \{u_k, u_{k+1}\}| = 2$. Since $J_1 \cup J_2$ is a maximal cell interval of $\mathcal{P}''$, then $f \in I_{\mathcal{P}''}$. The latter proves $I'' \subseteq I_{\mathcal{P}''} + (x_{u_1}, \dots, x_{u_r}, x_u)$. Similarly, the other inclusion follows, due to the fact that an inner interval in $\mathcal{P}''$ is either an inner interval of $\mathcal{P}_1$, of $\mathcal{P}_2$ (up to the translation defined in Definition 3.4), or it is contained in $J_1 \cup J_2$. Lastly, since $V(\mathcal{P}'') \cap F = \{v_k, v_{k+1}\}$, then

$$R/I'' \cong K[\mathcal{P}''] \otimes K[x_{v_1}, \dots, x_{v_{k-1}}, x_{v_{k+2}} \dots, x_{v_r}, x_v]$$

□

Corollary 3.11 *Let $\mathcal{P}$ be a simple thin polyomino, collapsible in I that has r cells, with $\mathcal{P}'$ and $\mathcal{P}''$ as in Definitions 3.3 and 3.4. Then,*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1}{1-t}\left(\mathrm{HP}_{K[\mathcal{P}']}(t) + \frac{t}{(1-t)^{r-1}} \cdot \mathrm{HP}_{K[\mathcal{P}'']}(t)\right)$$

Proof Let C be a leaf of I, and let u and v be the leaf corners of C with u satisfying the hypotheses of Lemma 3.10. We take the following short exact sequence:

$$0 \longrightarrow R/(I_{\mathcal{P}} : x_u) \longrightarrow R/I_{\mathcal{P}} \longrightarrow R/(I_{\mathcal{P}}, x_u) \longrightarrow 0$$

Since $\mathcal{P}$ is simple, then from Lemma 2.1 $I_{\mathcal{P}}$ is prime, that is $(I_{\mathcal{P}} : x_u) = I_{\mathcal{P}}$. Therefore, by Proposition 2.3.(2) we have

$$\mathrm{HP}_{R/I_{\mathcal{P}}}(t) = \frac{1}{1-t}\mathrm{HP}_{R/(I_{\mathcal{P}},x_u)}(t).$$

We study the Hilbert series of $R/(I_{\mathcal{P}}, x_u)$. By applying Proposition 2.3 to the following short exact sequence:

$$0 \longrightarrow R/((I_{\mathcal{P}}, x_u) : x_v) \longrightarrow R/(I_{\mathcal{P}}, x_u) \longrightarrow R/(I_{\mathcal{P}}, x_u, x_v) \longrightarrow 0$$

we get

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1}{1-t}\left(\mathrm{HP}_{R/(I_{\mathcal{P}},x_u,x_v)}(t) + t \cdot \mathrm{HP}_{R/((I_{\mathcal{P}},x_u):x_v)}(t)\right).$$

Furthermore, by Lemma 3.10, we have

1. $R/(I_{\mathcal{P}}, x_u, x_v) \cong K[\mathcal{P}']$;
2. $R/((I_{\mathcal{P}}, x_u) : x_v) \cong K[\mathcal{P}''] \otimes K[y_1, \dots, y_{r-1}]$.

It is well known that

$$\mathrm{HP}_{K[y_1,\ldots,y_n]}(t) = \frac{1}{(1-t)^n}$$

and

$$\mathrm{HP}_{A\otimes B}(t) = \mathrm{HP}_A(t)\cdot \mathrm{HP}_B(t),$$

that is

$$\mathrm{HP}_{R/((I_{\mathcal{P}},x_u):x_v)}(t) = \frac{1}{(1-t)^{r-1}}\cdot \mathrm{HP}_{K[\mathcal{P}'']}(t)$$

and the assertion follows. □

Let $\mathcal{P}$ be a cell interval with $\operatorname{rk}\mathcal{P} = r$. The ideal $I_{\mathcal{P}}$ can be seen as the determinantal ideal of a $2\times(r+1)$ matrix. The resolution of the above ideal is well known (see [3,5]), as well as its Hilbert series. For the sake of completeness, we give the following result

Lemma 3.12 *Let $\mathcal{P}$ be a cell interval with* $\operatorname{rk}\mathcal{P} = r$. *Then,*

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1+rt}{(1-t)^{r+2}}.$$

Proof By [5, Corollary 6.2], $I_{\mathcal{P}}$ has linear resolution, and $\beta_{i,i+1} = i\binom{r+1}{i+1}$ for $i = 1,\ldots,r$. It is well known that if M is an R-module, then

$$\mathrm{HP}_M(t) = \frac{1}{(1-t)^n}\sum_{i=0}^{n}\sum_{j\in\mathbb{Z}}(-1)^i\beta_{ij}t^j.$$

That is, the Hilbert series of $K[\mathcal{P}]$ is

$$\frac{1+\sum_{i=1}^{r-1}(-1)^i i\binom{r+1}{i+1}t^{i+1} + (-1)^r r t^{r+1}}{(1-t)^{2r+2}}. \qquad (*)$$

We study the coefficient $i\binom{r+1}{i+1}$ for $2\le i\le r-1$.

$$\begin{aligned} i\binom{r+1}{i+1} &= (i+1)\binom{r+1}{i+1} - \binom{r+1}{i+1} \\ &= (r+1)\binom{r}{i} - \binom{r+1}{i+1} = r\binom{r}{i} - \binom{r}{i+1}. \end{aligned}$$

Hence, the numerator of Equation (∗) becomes

$$\begin{aligned}
&1+\sum_{i=1}^{r-1}(-1)^i\left(r\binom{r}{i}-\binom{r}{i+1}\right)t^{i+1}+(-1)^r r t^{r+1}\\
&=1+\sum_{i=2}^{r}(-1)^i\binom{r}{i}t^i+\sum_{i=1}^{r}(-1)^i r\binom{r}{i}t^{i+1}-rt+rt\\
&=(1-t)^r+rt(1-t)^r.
\end{aligned}$$

That is,

$$\mathrm{HP}_{K[\mathcal{P}]}(t)=\frac{(1+rt)(1-t)^r}{(1-t)^{2r+2}},$$

and the assertion follows. □

We now state the main theorem (see also Examples 4.3 and 4.4).

Theorem 3.13 *Let $\mathcal{P}$ be a simple thin polyomino with*

$$\mathrm{HP}_{K[\mathcal{P}]}(t)=\frac{h(t)}{(1-t)^d}.$$

Then, $h(t)$ is the rook polynomial of $\mathcal{P}$.

Proof Let $I_1,\dots I_s$ be the maximal inner intervals of $\mathcal{P}$. We proceed by induction on $p=\operatorname{rk}\mathcal{P}$. If $p=1$, then $\mathcal{P}$ consists of one cell, and by Lemma 3.12, the statement follows. Let $p>1$ and assume the thesis true for any polyomino with rank less than or equal to $p-1$. If $s=1$, then $\mathcal{P}$ is a cell interval, and from Lemma 3.12, we have

$$\mathrm{HP}_{K[\mathcal{P}]}(t)=\frac{1+pt}{(1-t)^{p+2}}.$$

The polynomial $1+pt$ is the rook polynomial of a cell interval having p cells; that is, the assertion follows. If $s>1$, then $\mathcal{P}$ is not a cell interval; that is, from Proposition 3.7, $\mathcal{P}$ is collapsible in some maximal inner interval I. Assume that I has r cells. In order to apply Corollary 3.11, we focus on $\mathrm{HP}_{K[\mathcal{P}']}(t)$ and $\mathrm{HP}_{K[\mathcal{P}'']}(t)$. The polyomino $\mathcal{P}'$ has $p-1$ cells, while the polyomino $\mathcal{P}''$ has $p-r$ cells. Hence, from the inductive hypothesis we have

$$\mathrm{HP}_{K[\mathcal{P}']}(t)=\frac{\sum_{i=0}^{a}r'_i t^i}{(1-t)^{d_1}},$$

where $a = r(\mathcal{P})$ with $r'_a \geq 0$ due to Remark 3.5, and $\sum_{i=0}^{a} r'_i t^i$ is the rook polynomial of $\mathcal{P}'$, and

$$\mathrm{HP}_{K[\mathcal{P}'']}(t) = \frac{\sum_{i=0}^{b} r''_i t^i}{(1-t)^{d_2}}.$$

where $b = r(\mathcal{P}'') = r(\mathcal{P}) - 1$ due to Remark 3.5, and $\sum_{i=0}^{b} r''_i t^i$ is the rook polynomial of $\mathcal{P}''$. From Corollary 3.11, we get that $\mathrm{HP}_{K[\mathcal{P}]}(t)$ is equal to

$$\frac{1}{1-t}\left(\frac{\sum_{i=0}^{a} r'_i t^i}{(1-t)^{d_1}} + \frac{1}{(1-t)^{r-1}} \frac{\sum_{i=0}^{b} r''_i t^{i+1}}{(1-t)^{d_2}}\right) = \frac{\sum_{i=0}^{a} r'_i t^i}{(1-t)^{d_1+1}} + \frac{\sum_{i=0}^{b} r''_i t^{i+1}}{(1-t)^{d_2+r}}$$

We first show that $d_1 + 1 = d_2 + r = n - p$, where $n = |V(\mathcal{P})|$. Since $\mathcal{P}'$ is the polyomino having $n-2$ vertices and $p-1$ cells, then from Lemma 2.1, we have $(n-2) - (p-1) = n - p - 1$. Moreover, since I is on the $2r+2$ vertices $\{x_1, \dots, x_r, x, y_1, \dots, y_r, y\}$ but y_k, y_{k+1} for some k are corners of one cell of $\mathcal{P} \setminus I$, then $\mathcal{P}''$ is the polyomino having $n - 2r$ vertices and $p - r$ cells, hence from Lemma 2.1 $d_2 + r - 1 = (n-2r) - (p-r) + r - 1 = n - p - 1$. That is,

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1 + \sum_{i=1}^{r(\mathcal{P})} (r'_i + r''_{i-1}) t^i}{(1-t)^d}$$

For $1 \leq i \leq r(\mathcal{P})$, $r_i = r'_i + r''_{i-1}$. In fact, r_i is the number of ways of placing i non-attacking rooks on all of the cells of $\mathcal{P}$, whereas r'_i is the number of ways of placing i non-attacking rooks on the simple thin polyomino $\mathcal{P}'$, namely the number of ways of placing i non-attacking rooks on the cells $D \neq C$ of $\mathcal{P}$, and r''_{i-1} is the number of ways of placing $i-1$ non-attacking rooks on the simple thin polyomino $\mathcal{P}''$, namely the number of ways of placing $i-1$ non-attacking rooks on the cells D of $\mathcal{P}$ such that $D \notin I$, given that the i-th rook is placed on the cell C; hence, the thesis follows. □

We immediately deduce the following

Corollary 3.14 *Let $\mathcal{P}$ be a simple thin polyomino. Then, the Castelnuovo–Mumford regularity is $r(\mathcal{P})$ and the multiplicity of $K[\mathcal{P}]$ is $r_{\mathcal{P}}(1)$.*

Remark 3.15 In general, the equality $h(t) = r_{\mathcal{P}}(t)$ does not hold for any simple polyomino $\mathcal{P}$. Let $\mathcal{P}$ be the square tetromino. Then, $K[\mathcal{P}]$ is the toric ring related to the complete bipartite graph $K_{3,3}$, and from [19, Lemma 2.2], we have

$$h(t) = 1 + 4t + t^2 \text{ and } r_{\mathcal{P}}(t) = 1 + 4t + 2t^2.$$

Even though the two polynomials are different, they have the same degree, that is reg $K[\mathcal{P}] = r(\mathcal{P})$ also in this case.

4 Gorenstein simple thin polyominoes

In this section, we characterize the Gorenstein simple thin polyominoes. We start with a fundamental definition for our goal.

Definition 4.1 Let $\mathcal{P}$ be a simple thin polyomino. A cell C of $\mathcal{P}$ is *single* if there exists a unique maximal inner interval of $\mathcal{P}$ containing C. If any maximal inner interval of $\mathcal{P}$ has exactly one single cell, we say that $\mathcal{P}$ has the *S-property*.

Let $\mathcal{C}$ be the set of the single cells of a simple thin polyomino. We set $\mathcal{D}$ as the collection of cells $\mathcal{P} \setminus \mathcal{C}$. In particular since $\mathcal{P}$ is thin, then any cell of $\mathcal{D}$ belongs exactly to two maximal inner intervals of $\mathcal{P}$.

Theorem 4.2 *Let $\mathcal{P}$ be a simple thin polyomino, $I_1, \dots, I_s$ be its maximal inner intervals, and let $r_{\mathcal{P}}(t) = \sum_{k=0}^{s} r_k t^k$ be its rook polynomial. Then, the following conditions are equivalent:*

(a) *$K[\mathcal{P}]$ is Gorenstein;*
(b) *$\forall i = 0, \dots, s$ we have $r_i = r_{s-i}$;*
(c) *$\mathcal{P}$ satisfies the S-property.*

Proof (a)⇔(b): By combining Theorems 2.4 and 3.13, for a simple thin polyomino $\mathcal{P}$, the Cohen–Macaulay domain $K[\mathcal{P}] = R/I_{\mathcal{P}}$ is Gorenstein if and only if $\forall i = 0, \dots, s$, we have $r_i = r_{s-i}$, and the assertion follows. (c)⇒(b): Since $\mathcal{P}$ satisfies the S-property, then any maximal inner interval I of $\mathcal{P}$ contains a unique single cell C. Therefore, let $\mathcal{C} = \{C_1, \dots C_s\}$ be the set of the single cells of $\mathcal{P}$, and let $I_1, \dots, I_s$ be the maximal inner intervals of $\mathcal{P}$ such that $C_i \in I_i$. We set $\mathcal{D} = \mathcal{P} \setminus \mathcal{C}$. As we have observed above, any cell of $\mathcal{D}$ is the intersection of two maximal inner intervals of $\mathcal{P}$, and we denote by D_{jk} the cell of $\mathcal{D}$ in the intersection of I_j and I_k.

Let $\mathbf{i}$ be a subset of $[s] = \{1, 2, \dots, s\}$ of cardinality l, and let $\mathbf{jk} = \{\{j_1, k_1\}, \dots, \{j_m, k_m\}\}$ with $j_t, k_t \in [s]$ for $1 \leq t \leq m$. We denote by $\mathcal{C}_{\mathbf{i}} = \{C_i \in \mathcal{C} : i \in \mathbf{i}\}$ and by $\mathcal{D}_{\mathbf{jk}} = \{D_{jk} \in \mathcal{D} : \{j, k\} \in \mathbf{jk}\}$.

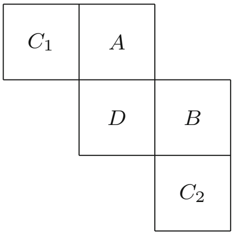
Let $\mathbf{j} = \{j_1, \dots j_m\}$ and $\mathbf{k} = \{k_1, \dots k_m\}$ be such that $\mathbf{j} \cap \mathbf{k} = \varnothing$, and let $\mathbf{i}$ be such that $\mathbf{i} \cap (\mathbf{j} \sqcup \mathbf{k}) = \varnothing$, then

$$\mathcal{C}_{\mathbf{i}} \cup \mathcal{D}_{\mathbf{jk}} \tag{1}$$

induces a set of $d = l + m$ non-attacking rooks, and any set of non-attacking rooks of cardinality d can be written in the form (1), and this configuration is unique because a set $\mathbf{jk}$ identifies a unique subset of $\mathcal{D}$ and thanks to the S-property, a set $\mathbf{i} \subset [\mathbf{s}]$ identifies a unique subset of $\mathcal{C}$. Our goal is to prove that for any configuration (1) of d non-attacking rooks, there exists a unique configuration of the form (1) of $s - d$ non-attacking rooks. Let $\overline{\mathcal{C}}_{\mathbf{i} \cup \mathbf{i} \cup \mathbf{k}} = \mathcal{C} \setminus (\mathcal{C}_{\mathbf{i}} \cup \mathcal{C}_{\mathbf{j}} \cup \mathcal{C}_{\mathbf{k}})$, and since $\mathbf{i} \cap (\mathbf{j} \cup \mathbf{k}) = \varnothing$, then $|\overline{\mathcal{C}}_{\mathbf{i} \cup \mathbf{j} \cup \mathbf{k}}| = s - (l + 2m)$. From the configuration of cardinality d in (1), we retrieve the following configuration of cardinality $s - d$,

$$\overline{\mathcal{C}}_{\mathbf{i} \cup \mathbf{j} \cup \mathbf{k}} \cup \mathcal{D}_{\mathbf{jk}}. \tag{2}$$

Fig. 5 A simple thin polyomino $\mathcal{Q}$ that does not satisfy the S-property



In fact, $s-(l+2m)+m=s-d$ and the configuration (2) satisfies the properties of configuration (1), and the configuration (2) is uniquely determined by (1) because $\mathcal{D}_{\mathbf{jk}}$ is fixed, and once we set $\mathcal{C}_{\mathbf{i}}$ and $\mathbf{j} \cup \mathbf{k}$, the complement set $\overline{\mathcal{C}}_{\mathbf{i}\cup\mathbf{j}\cup\mathbf{k}}$ is unique.

(b)$\Rightarrow$(c): By contraposition, assume that $\mathcal{P}$ does not satisfy the S-property, that is there exists an inner interval I of $\mathcal{P}$ having q single cells with $q \neq 1$. We want to prove that either $r_s > r_0 = 1$ or $r_{s-1} > r_1 = \operatorname{rk} \mathcal{P}$.

Let $q > 1$, and let C, C' be two single cells of I. Any set $\mathcal{C}$ of s non-attacking rooks contains a single cell C'' of I such that either $C'' \neq C$ or $C'' \neq C'$. In both cases, the sets $\mathcal{C} \setminus \{C''\} \cup C$ and $\mathcal{C} \setminus \{C''\} \cup C'$ are two distinct sets of s non-attacking rooks, that is $r_s > 1$, and it is a contradiction. Hence, from now on we assume that $\mathcal{P}$ do not exist maximal inner intervals with two or more single cells. That is, any maximal inner interval of $\mathcal{P}$ has either 0 or 1 single cells and in particular, we assume $q = 0$. Let $\mathcal{C}$ be a set of s non-attacking rooks of $\mathcal{P}$. In this case, one of the following is true:

1. any inner interval J intersecting I in a cell D contains a cell $C \neq D$ such that $C \in \mathcal{C}$, in particular $I \cap \mathcal{C} = \varnothing$;
2. there exists an inner interval J intersecting I in a cell $D \in \mathcal{C}$.

In case (1), $(\mathcal{C} \setminus \{C\}) \cup \{D\}$ is a set of s non-attacking rooks different from $\mathcal{C}$, that is $r_s > 1$, and it is a contradiction.

In case (2), we want to show $r_{s-1} > r_1$. Let E be a cell of $\mathcal{P}$. If $E \in \mathcal{C}$, then $\mathcal{C} \setminus \{E\}$ is a set of $s-1$ non-attacking rooks. If $E \notin \mathcal{C}$, then E is not single; that is, E is the intersection of two cell intervals I_1 and I_2. From the maximality of $\mathcal{C}$, there exist two cells $F \in I_1$ and $G \in I_2$ with $F, G \in \mathcal{C}$, and $\mathcal{C} \setminus \{F, G\} \cup \{E\}$ is a set of $s-1$ non-attacking rooks. Hence, $r_{s-1} \geq r_1$.

The hypothesis (2) implies that there exist some cells A, B, C_1, C_2 of $\mathcal{P}$ such that the polyomino $\mathcal{Q}$ in Fig. 5 is a subpolyomino of $\mathcal{P}$ (up to rotations and reflections). In fact, without loss of generality assume that A is a cell of I and B is a cell of J. Since I has no single cells, there exists an inner interval J' intersecting I in A. Moreover, if the cell B is single, then $B \in \mathcal{C}$ and this contradicts (2). Hence, there exists an inner interval J'' intersecting J in B.

Let F and G be the cells of $\mathcal{C}$ that belong to J' and J'', respectively. We consider the following sets of $s-1$ non-attacking rooks:

$$\mathcal{C} \setminus \{F, D\} \cup \{A\}, \mathcal{C} \setminus \{G, D\} \cup \{B\},\ \mathcal{C} \setminus \{F, G, D\} \cup \{A, B\}.$$

The first two were mentioned in the discussion above, while the third one increases the number r_{s-1}. Hence, $r_{s-1} > r_1$ that is a contradiction.

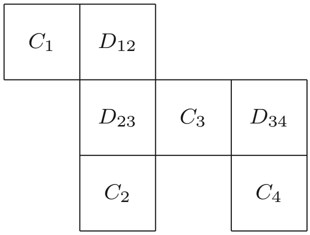


Fig. 6 A simple thin polyomino satisfying the S-property

□

Example 4.3 Let $\mathcal{P}$ be the polyomino in Fig. 6.

We see that $\mathcal{P}$ has 4 maximal inner intervals and a single cell for any of these ones; that is, $\mathcal{P}$ satisfies the S-property. We want to compute the Hilbert series of $K[\mathcal{P}]$. It is easy to see that $r(\mathcal{P}) = 4$. According to Theorem 3.13, the Hilbert series of $K[\mathcal{P}]$ is

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{\sum_{i=0}^{4} r_i t^i}{(1-t)^d}$$

where $d = |V(\mathcal{P})| - \mathrm{rk}\,\mathcal{P} = 16 - 7 = 9$. We compute r_i, namely the number of sets of i non-attacking rooks for $i = 0, \dots, 4$.

- $i = 0$. $\emptyset$;
- $i = 1$. $\{C_1\}, \{C_2\}, \{C_3\}, \{C_4\}, \{D_{12}\}, \{D_{23}\}, \{D_{34}\}$;
- $i = 2$. $\{C_1, D_{23}\}, \{C_1, C_2\}, \{C_1, C_3\}, \{C_1, D_{34}\}, \{C_1, C_4\}, \{D_{12}, C_3\}, \{D_{12}, D_{34}\}, \{D_{12}, C_4\}, \{C_2, C_3\}, \{C_2, D_{34}\}, \{C_2, C_4\}, \{D_{23}, C_4\}, \{C_3, C_4\}$;
- $i = 3$. $\{C_1, C_2, C_3\}, \{C_1, C_2, C_4\}, \{C_1, C_3, C_4\}, \{C_2, C_3, C_4\}, \{C_1, C_2, D_{34}\}, \{C_1, D_{23}, C_4\}$ $\{D_{12}, C_3, C_4\}$;
- $i = 4$. $\{C_1, C_2, C_3, C_4\}$.

It follows

$$r_0 = 1,\ r_1 = 7,\ r_2 = 13,\ r_3 = 7,\ r_4 = 1,$$

that is

$$\mathrm{HP}_{K[\mathcal{P}]}(t) = \frac{1 + 7t + 13t^2 + 7t^3 + t^4}{(1-t)^9}$$

and according to Theorem 2.4, $K[\mathcal{P}]$ is Gorenstein.

Example 4.4 In the notation of Theorem 4.2, we highlight that the condition $r_s = 1$ is not sufficient to guarantee that the polynomial has symmetric coefficients. In fact, let us consider the polyomino $\mathcal{Q}$ in Fig. 5. The rook number of $\mathcal{Q}$ is 3, and the rook polynomial of $\mathcal{Q}$ is

$$1 + 5t + 6t^2 + t^3;$$

in fact, the sets of i non-attacking rooks are

- $i = 0$. $\emptyset$;
- $i = 1$. $\{A\}, \{B\}, \{C_1\}, \{D\}, \{C_2\}$;
- $i = 2$. $\{C_1, D\}, \{C_1, C_2\}, \{D, C_2\}, \{B, C_1\}, \{A, C_2\}, \{A, B\}$;
- $i = 3$. $\{C_1, D, C_2\}$;

As already noted in the proof of Theorem 4.2 the fact that $r_2 > r_1$ depends on the set $\{A, B\}$.

To conclude the paper, we want to remark that among the thin polyominoes that are not simple, namely multiply connected, there are some non-prime ones, so that we cannot directly retrieve the Cohen–Macaulayness of $K[\mathcal{P}]$. Nevertheless, due to Theorem 3.13 and Remark 3.15, we conjecture the following.

Conjecture 4.5 *Let $\mathcal{P}$ be a polyomino. Then, $\mathcal{P}$ is thin if and only if $r_{\mathcal{P}}(t) = h(t)$.*

Moreover, due to Theorem 3.13 and [4, Theorem 2.3], we ask the following.

Question 4.6 *Let $\mathcal{P}$ be a polyomino. Then,* $\operatorname{reg} K[\mathcal{P}] = r(\mathcal{P})$*?*

References

1. Andrei, C.: Properties of the coordinate ring of a convex polyomino, preprint arXiv:1803.03801, 1–28, (2018)
2. Biermann, J., O'Keefe, A., Van Tuyl, A.: Bounds on the regularity of toric ideals of graphs. Adv. Appl. Math. **85**, 84–102 (2017)
3. Bruns, W., Vetter, U.: Determinantal Rings. Lecture Notes in Mathematics, 1327. Springer (1988)
4. Ene, V., Herzog, J., Qureshi, A.A., Romeo, F.: Regularity and the Gorenstein property of L-convex polyominoes, preprint arXiv:1911.08189, pp. 1–21, (2019)
5. Eisenbud, D.: The Geometry of Syzygies, Graduate Texts in Mathematics. Springer, Berlin (2005)
6. Golomb, S.W.: Polyominoes, Puzzles, Patterns, Problems, and Packagings, 2nd edn. Princeton University Press, Princeton (1994)
7. Herzog, J., Hibi, T., Ohsugi, H.: Binomial Ideals, Graduate Texts in Mathematics, vol. 279. Springer, Cham (2018)
8. Herzog, J., Qureshi, A.A., Shikama, A.: Gröbner bases of balanced polyominoes. Math. Nachr. **288**, 775–783 (2015)
9. Herzog, J., Saeedi Madani, S.: The coordinate ring of a simple polyomino. Illinois J. Math. **58**, 981–995 (2014)
10. Hibi, T., Ohsugi, H.: Special simplices and Gorenstein toric rings. J. Combin. Theory Ser. A **113**, 718–725 (2006)
11. Mascia, C., Rinaldo, G., Romeo, F.: Primality of multiply connected polyominoes. Accepted in Illinois J. Math. (2020)
12. Mascia, C., Rinaldo, G., Romeo, F.: *Primality of polyomino ideals by quadratic Gröbner basis*, preprint arXiv: 2005.08758, pp. 1–16, (2020)
13. Qureshi, A.A.: Ideals generated by 2-minors, collections of cells and stack polyominoes. J. Algebra **357**, 279–303 (2012)
14. Qureshi, A.A., Shibuta, T., Shikama, A.: Simple polyominoes are prime. J. Commut. Algebra **9**, 413–422 (2017)
15. Riordan, J.: An Introduction to Combinatorial Analysis, Wiley Publications in Mathematical Statistics. Wiley, Chapman & Hall Ltd., New York, London (1958)
16. Shikama, A.: Toric representation of algebras defined by certain nonsimple polyominoes. J. Commut. Algebra **10**, 265–274 (2018)
17. Stanley, R.: Hilbert functions of graded algebras. Adv. Math. **28**, 57–83 (1978)

18. Villarreal, R.: Monomial Algebras, 2nd edn. Taylor and Francis, CRC Press, Boca Raton (2015)
19. Villarreal,R.: Rees algebras of complete bipartite graphs. In: Ferrero, M., Doering, A.M.S. (eds.), XV Escola de Álgebra, Canela, July 1998-Rio Grande do Sul, Mat. Contemp., vol. 16 (1999)

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