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Primality of polyomino ideals by quadratic Gröbner basis

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Abstract
In this work, we provide a necessary and sufficient condition on a polyomino ideal for having the set of inner 2-minors as graded reverse lexicographic Gröbner basis, due to combinatorial properties of the polyomino itself. Moreover, we prove that when the latter holds the polyomino ideal coincides with the lattice ideal associated to the polyomino, that is the polyomino ideal is prime. As an application, we describe two new infinite families of prime polyominoes.

KEYWORDS
Gröbner basis, polyomino ideals

MSC (2020)
05E40, 13C05, 13P10

1 | INTRODUCTION

The ideals generated by a subset of t-minors of an $m \times n$ matrix of indeterminates are an intensively-studied class of binomial ideals, due to their applications in algebraic statistics. Among these ideals, one finds the determinantal ideals, see, for instance, [1] and its references to original articles, the ladder ideals introduced by Conca in [3], and the ideals of adjacent minors introduced by Hoşten and Sullivan in [9]. In 2012, a new class of ideals generated by 2-minors were defined by Qureshi in [12]: the *polyomino ideals*. They arise from two-dimensional objects obtained by joining edge by edge unitary squares, called polyominoes. Over the last few years, algebraic properties of polyomino ideals have been investigated, mainly exploiting the combinatorics of the underlying polyomino. One of the most challenging, and still unsolved, algebraic problems on polyominoes is the classification of the prime ones. The fact that a binomial ideal is a prime ideal if and only if it is a toric ideal explains the great interest in prime polyomino ideals. Several steps in this direction have been done, but giving a complete characterization of the prime polyomino ideals does not seem to be an easy task. In [6] and [13], the authors prove that simple polyominoes, namely without holes, are prime. In [8] and [14], a family of prime polyominoes obtained by removing a convex polyomino by a given rectangle was showed. In a more recent paper [10], it is demonstrated that if the polyomino $\mathcal{P}$ is prime, then it should have no zig-zag walks, and it is conjectured that this is also a sufficient condition for the primality of $\mathcal{P}$. This conjecture has been verified computationally for all the polyominoes of rank ≤ 14. Moreover, in the same work, the authors present a new infinite class of prime polyominoes: the grid polyominoes.

Beside the primality, another interesting question concerns the Gröbner basis of ideals generated by a subset of t-minors, see [11, 16] and [2]. As regards polyomino ideals, in [12], the author provides a necessary and sufficient condition for the set of inner 2-minors to be a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to two fixed lexicographic monomial orders. Whereas, in [7], Herzog, Qureshi and Shikama show that the ideal of a balanced polyomino has a quadratic Gröbner basis with respect to any monomial ideal, that is the ideal is radical.

In this work, we combine the two above-mentioned questions: we study the primality of the polyomino ideals, by computing their Gröbner basis with respect to particular graded reverse lexicographic monomial orders. In Section 2, we provide the basic definitions regarding polyominoes and their ideals of inner 2-minors. Moreover, we recall the definition given in [12] of the lattice ideal associated to a polyomino $\mathcal{P}$, and we show that it is the ideal quotient of the polyomino ideal $I_\mathcal{P}$ and a monomial. In Section 3, we define different graded reverse lexicographic monomial orders and, as in [12], we give a necessary and sufficient condition on $\mathcal{P}$ for having the set of inner 2-minors as reduced Gröbner basis of $I_\mathcal{P}$ (see Proposition 3.2). Starting from these monomial orders, for any corner v of the polyomino, we define new monomial orders $<_v$ such that the variable x_v is the smallest one with respect to $<_v$. We determine when $I_\mathcal{P}$ admits quadratic Gröbner basis with respect to $<_v$ (see Proposition 3.4). In this case, we prove that the ideal is prime (see Theorem 3.5). In the final section of this paper, we apply all the previous results on a class of polyominoes: the *thin polyominoes* (see Definition 4.1). We exhibit necessary and sufficient conditions in terms of the geometry of the thin polyomino so that its ideal has a quadratic Gröbner basis with respect to some graded reverse lexicographic monomial orders (see Theorem 4.4). As an application we find two subclasses of thin polyominoes that are prime (see Corollary 4.6 and 4.10): one is that of *thin cycles* (see Definition 4.5) with inner intervals of length at least 3, and the other consists of polyominoes obtained from grid polyominoes by the deletion of some cells, that we call *subgrid polyominoes* (see Definition 4.9).

2 | PRELIMINARIES AND LATTICE IDEALS

Let $a = (i, j), b = (k, \ell) \in \mathbb{N}^2$, with $i \le k$ and $j \le \ell$, the set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \le r \le k \text{ and } j \le s \le \ell\}$ is called an *interval* of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$. In particular, a, b are called *diagonal corners* and c, d *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the left lower (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a + (1, 0)$, $a + (0, 1), a + (1, 1)$. The sets $\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}$, and $\{a + (0, 1), a + (1, 1)\}$ are called the *edges* of C. Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected* if there is a sequence of cells $C = C_1, \dots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \dots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \dots, C_m$ is called a *path* (connecting C and D). A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P}) = \bigcup_{C\in\mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$.

A polyomino $\mathcal{Q}$ is said to be a *subpolyomino* of a polyomino $\mathcal{P}$ if each cell belonging to $\mathcal{Q}$ belongs to $\mathcal{P}$, and we write $\mathcal{Q} \subset \mathcal{P}$. A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$. We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \dots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \dots, m$.

A finite collection $\mathcal{H}$ of cells not in $\mathcal{P}$ is called a *hole* of $\mathcal{P}$ if any two cells in $\mathcal{H}$ are connected through a path of cells in $\mathcal{H}$, and $\mathcal{H}$ is maximal with respect to the inclusion. Note that a hole $\mathcal{H}$ of a polyomino $\mathcal{P}$ is itself a simple polyomino.

Let $\mathcal{P}$ be a polyomino. Let $\mathbb{K}$ be a field and $S = \mathbb{K}[x_v \mid v \in V(\mathcal{P})]$. The binomial $x_a x_b - x_c x_d \in S$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where c, d are the anti-diagonal corners of $[a, b]$. We denote by $\mathcal{M}$ the set of all inner 2-minors of $\mathcal{P}$. The ideal $I_\mathcal{P} \subset S$ generated by $\mathcal{M}$ is called the *polyomino ideal* of $\mathcal{P}$.

We recall that given a lattice $\Lambda \subseteq \mathbb{Z}^{m\times n}$, we attach a binomial ideal I_Λ called the *lattice ideal* of Λ such that

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_\Lambda \Leftrightarrow \mathbf{a} - \mathbf{b} \in \Lambda.$$

We say that a lattice Λ is *saturated* if for any $\mathbf{a} \in \mathbb{Z}^{m\times n}$, $c \in \mathbb{Z}$ such that $c\mathbf{a} \in \Lambda$, we have $\mathbf{a} \in \Lambda$. It is known that Λ is saturated if and only if I_Λ is prime. Let $\mathcal{P} \subseteq [(1, 1), (m, n)]$ be a polyomino. Let

$$\mathcal{B} = \left\{\mathbf{e}_{ij} \ : i \in \{1, \dots, m\},\ j \in \{1, \dots, n\}\right\}$$

be the canonical basis of $\mathbb{Z}^{m\times n}$ and let $C = \{C_1, \dots, C_r\}$ be the set of cells of $\mathcal{P}$. Let $\alpha : C \longrightarrow \mathbb{Z}^{m\times n}$ be such that $\alpha(C_k) = \mathbf{c}_k = \mathbf{e}_{ij} + \mathbf{e}_{i+1j+1} - \mathbf{e}_{i+1j} - \mathbf{e}_{ij+1}$, where (i, j) is the lower left corner of the cell C_k.

It is known from [4] that an ideal generated by any set of adjacent 2-minors of a $m \times n$ matrix is a lattice ideal and that its corresponding lattice is saturated. Hence, the lattice $\Lambda = \langle\{\mathbf{c}_k\}_{k=1,\dots,r}\rangle$ is a saturated lattice, and I_Λ is a prime ideal. In addition, it is known from [12] that for a collection $\mathcal{P}$ of cells of $\mathbb{N}^2$, $I_\mathcal{P}$ is prime if and only if $I_\mathcal{P} = I_\Lambda$. Moreover,

TABLE 1 Pairs of arrows that induce the total orders

$<^1$	$<^2$	$<^3$	$<^4$	$<^5$	$<^6$	$<^7$	$<^8$
$(\downarrow,\rightarrow)$	$(\downarrow,\leftarrow)$	$(\uparrow,\leftarrow)$	$(\uparrow,\rightarrow)$	$(\leftarrow,\uparrow)$	$(\rightarrow,\uparrow)$	$(\rightarrow,\downarrow)$	$(\leftarrow,\downarrow)$

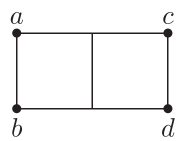


FIGURE 1 A rectangular polyomino $\mathcal{P}$

Lemma 2.1. *Let $\mathcal{P}$ be a collection of cells of $\mathbb{N}^2$, let S be the polynomial ring associated to $\mathcal{P}$. Then, there exists a monomial $u \in S$ such that*

$$I_\Lambda = (I_\mathcal{P} : u).$$

Proof. $\supseteq$). Let $u \in S$ be a monomial and let $f \in (I_\mathcal{P} : u)$. We have that $uf \in I_\mathcal{P} \subseteq I_\Lambda$. Since I_Λ is a prime ideal and $u \notin I_\Lambda$, then $f \in I_\Lambda$.

$\subseteq$). Let $f_\mathbf{e} = x^{\mathbf{e}^+} - x^{\mathbf{e}^-}$ be a generator of I_Λ, with

$$\mathbf{e} = \mathbf{e}^+ - \mathbf{e}^- = \sum_{k=1}^{r} \lambda_k \mathbf{c}_k = \sum_{k=1}^{r} \left((\lambda_k \mathbf{c}_k)^+ - (\lambda_k \mathbf{c}_k)^-\right) \in \Lambda,$$

where $\lambda_k \in \mathbb{Z}$, $\mathbf{v}^+$ denotes the vector obtained from $\mathbf{v} \in \mathbb{Z}^{m\times n}$ by replacing all negative components of $\mathbf{v}$ by zero, and $\mathbf{v}^- = -(\mathbf{v} - \mathbf{v}^+)$.

Let

$$\mathbf{v} = \sum_{k=1}^{r} (\lambda_k \mathbf{c}_k)^+ - \mathbf{e}^+ = \sum_{k=1}^{r} (\lambda_k \mathbf{c}_k)^- - \mathbf{e}^-.$$

We have that all the components of $\mathbf{v}$ are nonnegative, as for any $k \in \{1, \dots, r\}$ one has $\left(\mathbf{c}_k^+\right)_{ij} \geq (\mathbf{c}_k)_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This implies that the monomial $x^\mathbf{v} \in S$ is such that

$$x^\mathbf{v}\left(x^{\mathbf{e}^+} - x^{\mathbf{e}^-}\right) = \prod_{k=1}^{r} x^{(\lambda_k \mathbf{c}_k)^+} - \prod_{k=1}^{r} x^{(\lambda_k \mathbf{c}_k)^-} = \sum_{k=1}^{r} \mu_k \left(x^{\mathbf{c}_k^+} - x^{\mathbf{c}_k^-}\right) \in I_\mathcal{P},$$

for some $\mu_k \in S$. If we set u as the least common multiple of the elements $x^\mathbf{v}$ induced by all the generators $f_\mathbf{e}$ of I_Λ the assertion follows. $\square$

3 | QUADRATIC GRADED REVERSE LEXICOGRAPHIC GRÖBNER BASIS

Consider the total orders $<^i$, with $i \in \{1, \dots, 8\}$, on $\mathbb{N}^2$ induced by the pairs of arrows displayed in Table 1.

Given $a = (a_1, a_2)$ and $b = (b_1, b_2)$, the horizontal arrows refer to the first coordinates, a_1 and b_1, while the vertical ones to the second coordinates, a_2 and b_2. Each arrow goes from the minimum to the maximum. For any pair of arrows, that is for any total order, we first compare the coordinate given by the second arrow, and, if they are equal, then we compare the coordinates given by the first arrow. For instance, $a <^1 b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 > b_2$. That is, let $a, b, c, d \in V(\mathcal{P})$ be as in Figure 1.

Then it holds $a <^1 b <^1 c <^1 d$. The latter explains the order of the arrows, that is, we can order a set of vertices from the minimum to the maximum by firstly following the direction given by the first arrow and then the direction given by the second one. Similarly $(a_1, a_2) <^5 (b_1, b_2)$ if $a_2 < b_2$ or $a_2 = b_2$ and $a_1 > b_1$ and then one can recover all of the other orders. In the next remark, we show the relations between the orders $<^i$.

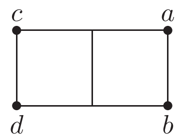


FIGURE 2 The polyomino $\mathcal{P}'$: the reflection of $\mathcal{P}$ with respect to $\{c, d\}$

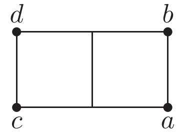


FIGURE 3 The polyomino $\mathcal{P}''$: the 180 degree rotation of $\mathcal{P}$

Remark 3.1. Let $\mathcal{P}$ be the polyomino in Figure 1. Then with respect to the orders $<^1$ induced by $(\downarrow, \rightarrow)$, $<^2$ induced by $(\downarrow, \leftarrow)$, $<^3$ induced by $(\uparrow, \leftarrow)$ we have

$$a <^1 b <^1 c <^1 d, \;\; c <^2 d <^2 a <^2 b, \;\; d <^3 c <^3 b <^3 a.$$

Let $\mathcal{P}'$ and $\mathcal{P}''$ be respectively the reflection of $\mathcal{P}$ with respect to the line containing the edge $\{c, d\}$ (Figure 2) and the 180 degree rotation of $\mathcal{P}$ (Figure 3).

We observe that in $\mathcal{P}'$ we have $c <^1 d <^1 a <^1 b$.

We observe that in $\mathcal{P}''$ we have $d <^1 c <^1 b <^1 a$. We conclude that the order $<^2$ it is equal to the order $<^1$ up to a reflection of the polyomino, while the order $<^3$ is equal to the order $<^1$ up to a 180 degree rotation of the polyomino. Similarly the other relations follow.

The total orders $<^i$, with $i \in \{1, \dots, 8\}$, on the vertices of $\mathcal{P}$ induce in a natural way the graded reverse lexicographic monomial orders $<^i_{\text{grevlex}}$, with $i \in \{1, \dots, 8\}$, on $S = \mathbb{K}[x_v \mid v \in V(\mathcal{P})]$, respectively.

As in [12, Theorem 4.1], the next proposition gives a necessary and sufficient condition on $\mathcal{P}$ for having $\mathcal{M}$ as quadratic reduced Gröbner basis of $I_{\mathcal{P}}$.

From now on, we set $\mathcal{O} = \{1, 3, 5, 7\}$ and $\mathcal{E} = \{2, 4, 6, 8\}$.

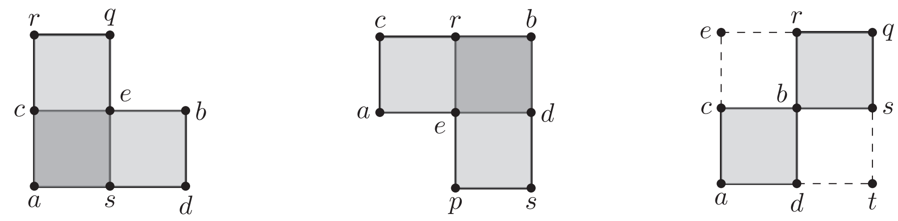
Proposition 3.2. *Let $\mathcal{P}$ be a polyomino. $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for $i \in \mathcal{O}$, if and only if for any two intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, at least one interval between $[a, f]$ and $[a, g]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b, e]$. Similarly, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for $i \in \mathcal{E}$, if and only if for any two inner intervals $[a, b]$ and $[e, f]$ of $\mathcal{P}$, with d anti-diagonal corner of both the inner intervals, either a, e or b, f are anti-diagonal corners of an inner interval of $\mathcal{P}$.*

Proof. We are going to prove the statement only for $<^1_{\text{grevlex}}$, then, by similar arguments and by Remark 3.1, the other cases follow. The others follow in a similar way. The set $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^1_{\text{grevlex}}$ if and only if all S-polynomials of inner 2-minors of $I_{\mathcal{P}}$ reduce to 0. Let $f, g \in \mathcal{M}$, where $f = x_a x_b - x_c x_d$ is associated to the inner interval $[a, b]$ of $\mathcal{P}$ and $g = x_p x_q - x_r x_s$ is associated to the inner interval $[p, q]$ of $\mathcal{P}$. In the following, we denote by S the S-polynomial between f and g and by $\text{in}(h)$ the leading monomial of a polynomial h. We consider the non-trivial cases when $\gcd(\text{in}(f), \text{in}(g)) \neq 1$. Moreover, if one of the inner intervals, namely $[a, b]$, is contained in the second one, namely $[p, q]$, S reduces to 0 since the polyomino ideal is generated by all inner 2-minors. In the following, denote by $<$ the total order $<^1$ on the vertices of $\mathcal{P}$. Without loss of generality, let $a \leq p$. Therefore, we have to consider the following cases: $a = p$, $b = q$, and $b = p$ (Figure 4).

Let $a = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_a x_q - x_r x_s$, and assume $r < c < a < q < s < b < d$ as in Figure 5. We have $S = x_q x_c x_d - x_b x_r x_s$ and $\text{in}(S) = x_q x_c x_d$. Since $\text{in}\left(f_{c,q}\right) = x_c x_q$, we get

$$S = x_d(x_c x_q - x_r x_e) - x_r(x_s x_b - x_e x_d),$$

that is S reduces to 0 with respect to $\mathcal{M}$.



(A) Case $a = p$ (B) Case $b = q$ (C) Case $b = p$

FIGURE 4 Cases to consider

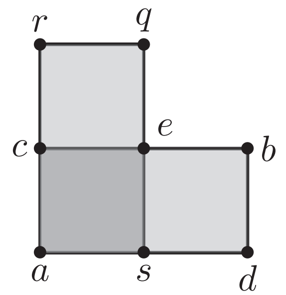


FIGURE 5 Case $a = p$

Let $b = q$, and assume $c < a < r < p < b < d < s$ as in Figure 3. We have $S = x_a x_r x_s - x_c x_d x_p$ and $\mathrm{in}(S) = x_a x_r x_s$. Since $\mathrm{in}(f_{a,r}) = x_a x_r$, we get

$$S = x_s(x_a x_r - x_c x_e) - x_c(x_p x_d - x_e x_s),$$

that is S reduces to 0 with respect to $\mathcal{M}$.

Let $b = p$, and assume $c < a < r < b < d < q < s$ as in Figure 3. We have $S = x_a x_r x_s - x_q x_c x_d$ and $\mathrm{in}(S) = x_a x_r x_s$. If neither $[a, s]$ nor $[a, r]$ is an inner interval of $\mathcal{P}$, then S does not reduce to 0 with respect to $\mathcal{M}$ and the Gröbner basis is not quadratic. Furthermore, if $[a, s]$ is an inner interval of $\mathcal{P}$, since $\mathrm{in}(f_{a,s}) = x_a x_s$, we get

$$S = x_r(x_a x_s - x_c x_t) - x_c(x_d x_q - x_r x_t).$$

If $[a, r]$ is an inner interval of $\mathcal{P}$, since $\mathrm{in}(f_{a,r}) = x_a x_r$, we get

$$S = x_s(x_a x_r - x_e x_d) - x_d(x_c x_q - x_e x_s).$$

It shows that in both situations S reduces to 0 with respect to $\mathcal{M}$. The latter shows that S reduces to 0 with respect to $\mathcal{M}$ if and only if either $[a, s]$ or $[a, r]$ is an inner interval of $\mathcal{P}$ and the thesis follows. $\square$

Let $V(\mathcal{P}) = \{v_1, \dots, v_n\}$. Given a monomial order $<$ such that we have

$$x_{v_1} < x_{v_2} < \cdots < x_{v_n},$$

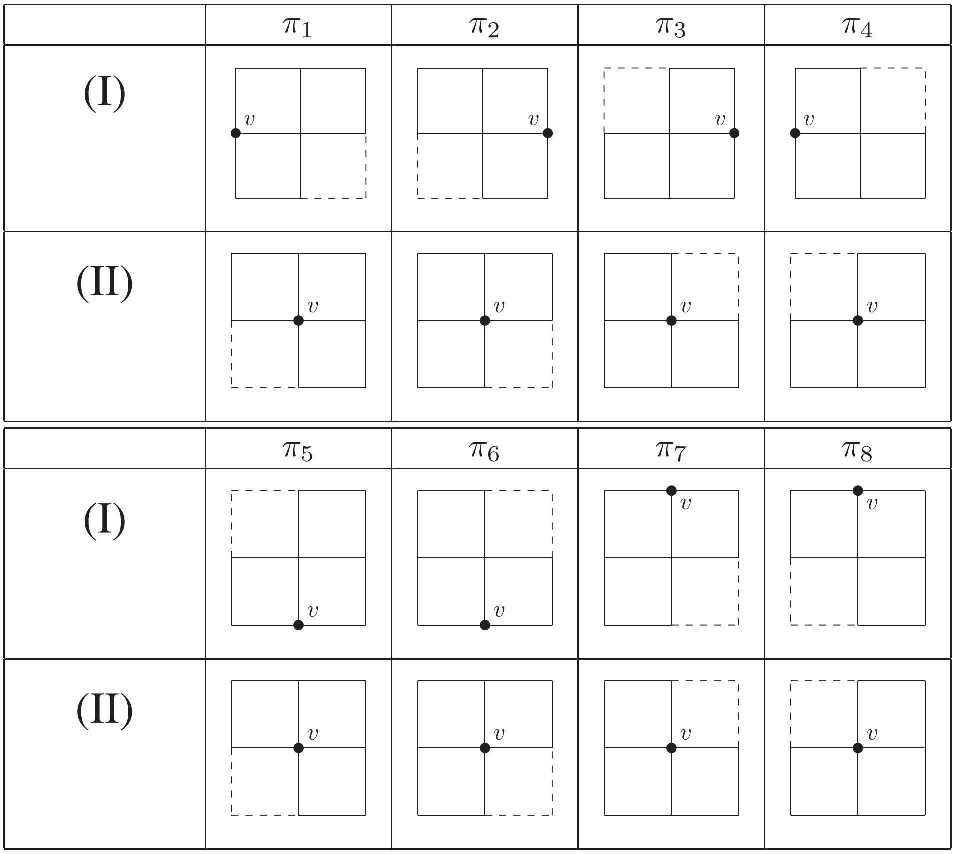
we define by $<_v$, with $v = v_k \in V(\mathcal{P})$, the following monomial order:

$$x_{v_k} < x_{v_{k+1}} < \cdots < x_{v_n} < x_{v_1} < x_{v_2} < \cdots < x_{v_{k-1}}.$$

From now on, we will denote $\left(<^i_{\mathrm{grevlex}} \right)_v$ by $<^i_v$, for any $i \in \{1, \dots, 8\}$.

Definition 3.3. Let $\mathcal{P}$ be a polyomino and let $v \in V(\mathcal{P})$. We say that v satisfies the condition π_1 if it fulfils at least one of the following conditions:

TABLE 2 Conditions π_i, for $i = 1, \dots, 8$



(I) There exist two inner intervals $I = [a, b]$ and $J = [b, q]$ of $\mathcal{P}$, with v upper left corner of I, and s the lower right corner of J, such that $[v, q]$ is inner interval of $\mathcal{P}$, whereas the interval $[a, s]$ is not (see Table 2, Case π_1 (I)).

(II) There exist two inner intervals $K = [a, b]$ and $L = [p, q]$, with v lower right corner of K and upper left corner of L, such that the interval having b and q as anti-diagonal corners is inner interval of $\mathcal{P}$, whereas the interval having a and p as anti-diagonal corners is not (see Table 2, Case π_1 (II)).

In a similar way, by Remark 3.1 and by using suitable rotations and/or reflections, one can define v satisfying the condition π_i, for $i \in \{2, \dots, 8\}$, if it fulfils at least one of the cases (I) and (II) displayed in Table 2.

Proposition 3.4. *Let $\mathcal{P}$ be a polyomino such that $I_{\mathcal{P}}$ has $\mathcal{M}$ as reduced Gröbner basis with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively). If $v \in V(\mathcal{P})$ does not satisfy π_k for some $k \in \mathcal{O}$ ($k \in \mathcal{E}$, respectively), then $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$.*

Proof. Assume that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. Let $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_r x_s$ be associated to the inner interval $[a, b]$ and $[p, q]$ of $\mathcal{P}$, respectively. Let $v \in V(\mathcal{P})$. We have to show that for each pair of inner 2-minors, f and g, the corresponding S-polynomial reduces to 0 with respect to a fixed monomial order $<^i_v$, with $i \in \mathcal{O}$. In the following, we denote by S the S-polynomial between f and g, by $\mathrm{in}(h)$ the leading monomial of a polynomial h, and by $f_{m,n}$ the inner 2-minor associated to the inner interval $[m, n]$ of $\mathcal{P}$.

We leave to the reader the trivial cases $\{a, b, c, d\} \cap \{p, q, r, s\} = \emptyset$, and $|\{a, b, c, d\} \cap \{p, q, r, s\}| = 2$ where S reduces to 0 since the polyomino ideal is generated by all inner 2-minors.

Note that if, for all vertices $w \in \{a, b, c, d, p, q, r, s\}$ and a monomial order $<^i_{\text{grevlex}}$, for some $i \in \mathcal{O}$, it holds $x_w <^i_v x_v$ or $x_v <^i_v x_w$, then S reduces to 0 with respect to $<^i_v$, since it reduces to 0 with respect to $<^i_{\text{grevlex}}$.

If one of the inner intervals, namely $[a, b]$, is contained in the second one, namely $[p, q]$, S reduces to 0 since the polyomino ideal is generated by all inner 2-minors. In the following, denote by $<$ the total order $<^1$ on the vertices of $\mathcal{P}$.

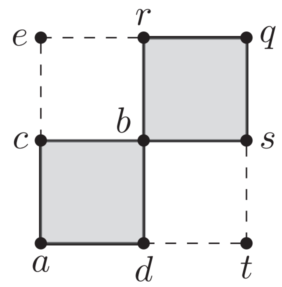


FIGURE 6 Case $b = p$

Without loss of generality, let $a \leq p$. Therefore, we have to consider the following cases:

$$a = p, \qquad b, d \in \{p, q, r, s\}, \qquad c \in \{p, r\}.$$

If v does not satisfy the condition π_k, for some $k \in \mathcal{O}$, we fix the monomial order $<_v^k$.

Assume $k = 1$.

Let $a = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_a x_q - x_r x_s$, and $r < c < a < q < s < b < d$ as in Figure 5.

We start by observing that if $r < v \leq b$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_r x_s x_b - x_c x_d x_q$. If $b < v \leq d$, then $\mathrm{in}(S) = x_r x_s x_b$. Since $\mathrm{in}(f_{s,b}) = x_s x_b$, then

$$S = x_r(x_s x_b - x_e x_d) - x_d(x_c x_q - x_r x_e),$$

that is S reduces to 0 with respect to the inner 2-minors $f_{s,b}$ and $f_{c,q}$. If $v = r$, then $\mathrm{in}(S) = x_c x_d x_q$. Since $\mathrm{in}(f_{c,q}) = x_c x_q$, then

$$S = -x_d(x_c x_q - x_r x_e) + x_r(x_s x_b - x_e x_d),$$

that is S reduces to 0 with respect to the inner 2-minors $f_{c,q}$ and $f_{s,b}$.

Let $b = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_b x_q - x_r x_s$, and$c < a < r < b < d < q < s$, as in Figure 6.

If $c < v \leq q$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_a x_r x_s - x_q x_c x_d$. If $q < v \leq s$, then $\mathrm{in}(S) = x_q x_c x_d$. By hypothesis, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\mathrm{grevlex}}$ with $i \in \mathcal{O}$, hence, from Proposition 3.2, either $[c, q]$ or $[d, q]$ is an inner interval of $\mathcal{P}$, with $\mathrm{in}(f_{c,q}) = x_c x_q$ and $\mathrm{in}(f_{d,q}) = x_d x_q$, and then

$$S = x_d(x_c x_q - x_e x_s) - x_s(x_a x_r - x_e x_d)$$

or

$$S = -x_c(x_d x_q - x_r x_t) + x_r(x_a x_s - x_c x_t),$$

that is S reduces to 0 with respect to the inner 2-minors either $f_{c,q}$ and $f_{a,r}$ or $f_{d,q}$ and $f_{a,s}$. If $v = c$, then $\mathrm{in}(S) = x_a x_r x_s$. By hypothesis, either $[a, r]$ or $[a, s]$ is an inner interval of $\mathcal{P}$, with $\mathrm{in}(f_{a,r}) = x_e x_d$ and $\mathrm{in}(f_{a,s}) = x_a x_s$. If $[a, r]$ is an inner interval of $\mathcal{P}$, but $[a, s]$ is not, then v satisfies the condition π_1, so we have not to consider this case. Whereas, if $[a, s]$ is an inner interval, since $\mathrm{in}(f_{a,s}) = x_a x_s$, then

$$S = x_r(x_a x_s - x_c x_t) - x_c(x_d x_q - x_r x_t),$$

it follows that S reduces to 0.

Note that when $v = c$, if $[a, r]$ is an inner interval of $\mathcal{P}$, but $[a, s]$ is not, that is v satisfies π_1, in particular the condition π_1 (I), then S does not reduce to 0 with respect to $\mathcal{M}$ and $<_v^1$. In fact, $\mathrm{in}(S) = x_a x_r x_s$, but the monomials $x_a x_r$, $x_a x_s$, and $x_r x_s$ are not leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies the hypothesis v not satisfying the condition π_1.

Let $b = r$, that is that is $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_b x_s$. We have to distinguish two different situations: $p < d$ (see Figure 7 (A)) or $p > d$ (see Figure 7 (B)).

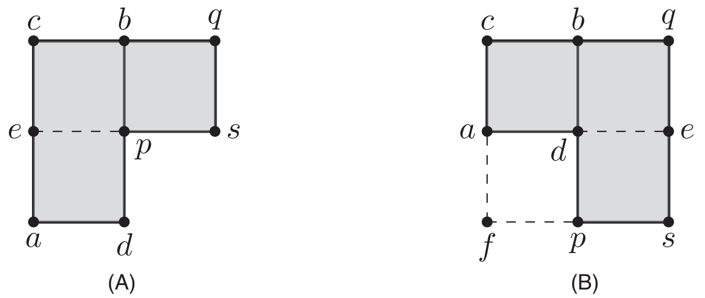


FIGURE 7 Case $b = r$

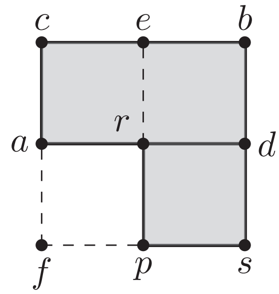


FIGURE 8 Case $d = q$

Assume $p < d$, then $c < a < b < p < d < q < s$, as in Figure 7 (A). If $c \le v \le b$ or $q < v \le s$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, $S = x_a x_p x_q - x_c x_d x_s$. If $b < v \le p$ or $d < v \le q$, then $\mathrm{in}(S) = x_c x_d x_s$ and $\mathrm{in}(f_{e,q}) = x_c x_s$. If $p < v \le d$, then $\mathrm{in}(S) = x_a x_p x_q$ and $\mathrm{in}(f_{a,p}) = x_a x_p$. Therefore,

$$S = x_d(x_e x_q - x_c x_s) + x_q(x_a x_p - x_e x_d),$$

that is S reduces to 0 in all of these cases.

Assume $p > d$, then $c < a < b < d < p < q < s$, as in Figure 7 (B). If $c \le v \le b$ or $q < v \le s$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_a x_p x_q - x_c x_d x_s$. If $b < v \le d$, then $\mathrm{in}(S) = x_a x_p x_q$. By hypothesis, v does not satisfy the condition π_1, hence $[f, d]$ is an inner interval of $\mathcal{P}$. Since $\mathrm{in}(f_{f,d}) = x_a x_p$, then

$$S = -x_q(x_f x_d - x_a x_p) + x_d(x_f x_q - x_c x_s),$$

that is S reduces to 0. If $d < v \le q$, then $\mathrm{in}(S) = x_c x_d x_s$. Since $\mathrm{in}(f_{p,e}) = x_d x_s$, it follows

$$S = x_c(x_p x_e - x_d x_s) + x_p(x_a x_q - x_c x_e),$$

that is S reduces to 0.

Note that when $b < v \le d$, if $[f, d]$ is not an inner interval of $\mathcal{P}$, then v satisfies π_1, in particular the condition π_1 (II). In this case, S does not reduce to 0 with respect to $\mathcal{M}$ and $<^1_v$. In fact, $\mathrm{in}(S) = x_a x_p x_q$, but the monomials $x_a x_p$, $x_a x_q$, and $x_p x_q$ are not leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies, once again, the hypothesis v not satisfying the condition π_1.

Let $d = q$, that is $f = x_a x_b - x_c x_d$ and $g = x_p x_d - x_r x_s$, and $c < a < r < p < b < d < s$, as showed in Figure 8.

If either $v = c$ or $r < v \le s$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_a x_b x_p - x_c x_r x_s$. If $c < v \le a$, then $\mathrm{in}(S) = x_c x_r x_s$. Since $\mathrm{in}(f_{c,r}) = x_c x_r$, then

$$S = x_s(x_a x_e - x_c x_r) + x_a(x_p x_b - x_s x_e),$$

that is S reduces to 0. If $a < v \le r$, then $\mathrm{in}(S) = x_a x_b x_p$. By hypothesis, v does not satisfy π_1, that is $[f, r]$ is an inner interval of $\mathcal{P}$. Since $\mathrm{in}\left(f_{f,r}\right) = x_a x_p$, then

$$S = -x_b\left(x_f x_r - x_a x_p\right) + x_r\left(x_f x_b - x_c x_s\right),$$

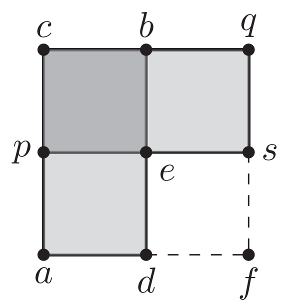


FIGURE 9 Case $c = r$

that is S reduces to 0.

Let $c = r$, that is $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_c x_s$, and $c < p < a < b < d < q < s$, as showed in Figure 9.

If either $v = c$ or $b < v \leq s$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, we have $S = x_a x_b x_s - x_d x_p x_q$. If $c < v \leq p$, $\text{in}(S) = x_a x_b x_s$. Since v does not satisfy π_1, then $[a, s]$ is an inner interval of $\mathcal{P}$ and $\text{in}(f_{a,s}) = x_a x_s$. Therefore,

$$S = x_b\left(x_a x_s - x_p x_f\right) - x_p\left(x_d x_q - x_b x_f\right),$$

that is S reduces to 0. If $p < v \leq b$, then $\text{in}(S) = x_d x_p x_q$ and $\text{in}(f_{a,e}) = x_p x_d$. Therefore,

$$S = x_q(x_a x_e - x_p x_d) - x_a(x_e x_q - x_b x_s),$$

that is S reduces to 0. For the sake of brevity, we leave to readers to check, in a similar way, that if $b \in \{q, s\}$, $d \in \{p, r, s\}$, and $c = p$, then all the S-polynomials reduce to 0. Moreover, for no one of the corners v in these cases it needs to require the hypothesis that v does not satisfy the condition π_1.

We have proved that when $k = 1$, all and only the cases in which S does not reduce to zero with respect to $\mathcal{M}$ and $<_v^1$ are when v satisfies π_1. Thanks to Remark 3.1. and Definition 3.3, for any $k \in \mathcal{O}$, all and only the cases in which S does not reduce to zero with respect to $<_v^k$ and $\mathcal{M}$ are when v satisfies π_k. Then, the statement holds for any $k \in \mathcal{O}$ and v not satisfying π_k. $\square$

We now prove the main theorem of this section.

Theorem 3.5. *Let $\mathcal{P}$ be a polyomino such that $I_{\mathcal{P}}$ has $\mathcal{M}$ as reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<_{\text{grevlex}}^i$, with $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively). If, for all $v \in V(\mathcal{P})$, there exists a $k_v \in \mathcal{O}$ ($k_v \in \mathcal{E}$, respectively) such that v does not satisfy π_{k_v}, then*

1. *$\mathcal{M}$ forms a reduced Gröbner basis with respect to $<_v^{k_v}$, for all $v \in V(\mathcal{P})$;*
2. *$I_{\mathcal{P}}$ is prime.*

Proof.

(1) It is an immediate consequence of Proposition 3.4.

(2) Fix $v \in V(\mathcal{P})$. By (1), let $<_v$ denote the monomial order for which $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$. By [15, Lemma 12.1], the reduced Gröbner basis of $(I_{\mathcal{P}} : x_v)$ with respect to $<_v$ is given by

$$\{f \in \mathcal{M} \mid x_v \text{ does not divide } f\} \cup \{f/x_v \mid f \in \mathcal{M} \text{ and } x_v \text{ divides } f\}.$$

Since all $f \in \mathcal{M}$ are not divisible by x_v, the reduced Gröbner basis of $(I_{\mathcal{P}} : x_v)$ with respect to $<_v$ is $\mathcal{M}$. Therefore $(I_{\mathcal{P}} : x_v) = I_{\mathcal{P}}$, for all $x_v \in V(\mathcal{P})$. It follows that $(I_{\mathcal{P}} : u) = I_{\mathcal{P}}$ for any monomial $u \in S$. By Lemma 2.1, we have that there exists a monomial $u \in S$ such that $I_\Lambda = (I_{\mathcal{P}} : u)$. Then

$$I_\Lambda = (I_{\mathcal{P}} : u) = I_{\mathcal{P}}.$$

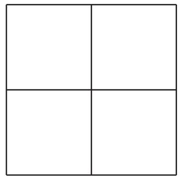


FIGURE 10 The polyomino $\mathcal{Q}$

It follows that $I_{\mathcal{P}}$ coincides with the lattice ideal I_Λ, which is prime. Therefore, $I_{\mathcal{P}}$ is a prime ideal, as well. $\square$

4 | THIN POLYOMINOES

In this section, we introduce the class of thin polyominoes and we rephrase the geometric condition for the quadratic Gröbner basis of $I_{\mathcal{P}}$ in Proposition 3.2 in terms of some subpolyominoes of the thin polyomino $\mathcal{P}$. Thanks to the above interpretation, we find two new classes of thin polyominoes having a prime polyomino ideal: the thin cycle with no maximal inner interval of length 2 and the subgrid polyominoes.

Definition 4.1. Let $\mathcal{P}$ be a polyomino. We say that $\mathcal{P}$ is *thin* if $\mathcal{P}$ does not have the polyomino $\mathcal{Q}$ in Figure 10 as a subpolyomino.

Theorem 4.2. *Let $\mathcal{P}$ be a thin polyomino such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$ (for $i \in \mathcal{E}$, respectively). Then, for any $v \in V(\mathcal{P})$, there exists $k \in \mathcal{O}$ ($k \in \mathcal{E}$, respectively) such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$.*

Proof. Assume that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. Let $v \in V(\mathcal{P})$. From Proposition 3.4 it suffices to show that there exists $k \in \mathcal{O}$ such that v does not satisfy π_k.

We claim that v can not satisfy simultaneously π_1 and π_3. In fact, if v satisfies simultaneously π_1 and π_3, then there exist four cells C, D, E, F of $\mathcal{P}$ such that $C \cap D \cap E \cap F = \{v\}$. From Table 2, if v satisfies π_1 then there exist two cells C, D of $\mathcal{P}$ such that v is simultaneously the lower left corner of C and the upper left corner of D, while if $\mathcal{P}$ satisfies π_3 then there exist two cells E, F of $\mathcal{P}$ such that v is simultaneously the lower right corner of E and the upper right corner of F. Since v satisfies simultaneously π_1 and π_3, the cells C, D, E, F are the ones desired. This implies that the polyomino $\mathcal{Q}$ in Figure 10 is a subpolyomino of $\mathcal{P}$ and then $\mathcal{P}$ is not thin, which is a contradiction. It follows that there exists at least a $k \in \mathcal{O}$ such that v does not satisfy π_k, as desired. $\square$

Corollary 4.3. *Let $\mathcal{P}$ be a thin polyomino such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \{1, \dots, 8\}$. Then $I_{\mathcal{P}}$ is prime.*

Proof. By Theorem 4.2, for any $v \in V(\mathcal{P})$, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$, for some $k \in \{1, \dots, 8\}$. By Theorem 3.5, it follows that $I_{\mathcal{P}}$ is prime. $\square$

Theorem 4.4. *Let $\mathcal{P}$ be a thin polyomino. The following facts are equivalent:*

1. *$\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively);*
2. *there are no cells $C, D \notin \mathcal{P}$ and $E, F \in \mathcal{P}$ such that $C \cap D \cap E \cap F \neq \emptyset$ as in Figure 11 (a) (Figure 11(b), respectively) and the polyominoes in Figure 12 (i) and (ii) (in Figure 12 (iii) and (iv), respectively) are not subpolyominoes of $\mathcal{P}$.*

Proof. We prove the equivalent statements for $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. The case $<^i_{\text{grevlex}}$ for $i \in \mathcal{E}$ can be done similarly.

(1) ⇒ (2). Firstly, let E, F be two cells of $\mathcal{P}$ as in Figure 11(a). Since, by hypothesis, $\mathcal{M}$ is a quadratic Gröbner basis, by Proposition 3.2, at least one cell between C and D must be a cell of P. That is the situation displayed in Figure 11(a) is not possible. Secondly, assume, by contradiction, that the polyominoes in Figure 12(i) and (ii) are subpolyominoes of $\mathcal{P}$. Then we consider the inner intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$ as in Figure 13.

FIGURE 11 Configurations of cells to avoid for having quadratic Gröbner bases

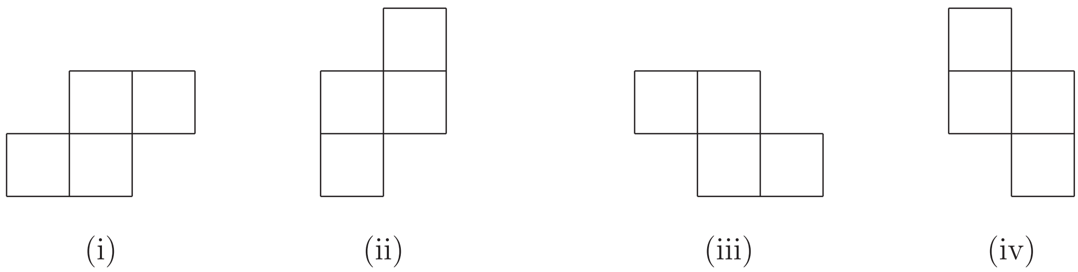


FIGURE 12 Subpolyominoes to avoid for having quadratic Gröbner bases

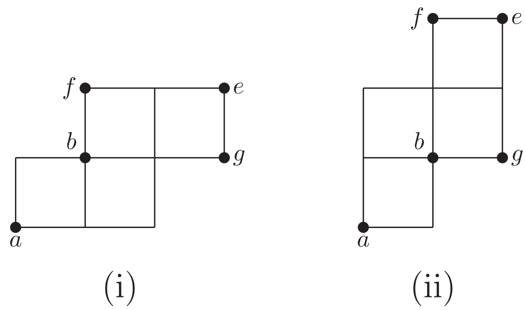


FIGURE 13 Possible positions for the inner intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$

By Proposition 3.2, at least one between $[a, g]$ and $[a, f]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b, e]$. In both cases, we get a polyomino that is not thin, which is a contradiction.

(2) ⇒ (1). Assume, by contradiction, that $\mathcal{M}$ does not form a quadratic Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$. According to Proposition 2.1, there exist two inner intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, where $[a, b]$ has anti-diagonal corners c and d, and $[b, e]$ has anti-diagonal corners f and g, such that neither $[a, f]$ nor $[a, g]$ is an inner interval of $\mathcal{P}$. Let E and F be respectively cells of $[a, b]$ and $[b, e]$ such that $E \cap F = \{b\}$. Let C and D be respectively cells of $[a, f]$ and $[a, g]$ such that $E \cap C \cap D \cap F = \{b\}$. Since $\mathcal{P}$ is thin, the cells C and D can not simultaneously be cells of $\mathcal{P}$. If neither C nor D is a cell of $\mathcal{P}$, then C, D, E, and F are cells as in Figure 11(a) and this is a contradiction. Assume, without loss of generality, that $C \notin \mathcal{P}$, but $D \in \mathcal{P}$. Since $[a, g]$ is not an inner interval of $\mathcal{P}$, then d and g are not both corners of D.

Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ given by the union of the cells of $[a, b]$, $[b, e]$ and D, as in Figure 14. Then, one of the two subpolyominoes displayed in Figure 12(i) and (ii) is a subpolyomino of $\mathcal{P}'$, and then of $\mathcal{P}$, which is a contradiction. □

Definition 4.5. Let $\mathcal{P} = \{C_1, \dots, C_n\}$ be a thin polyomino. If there exists a relabelling of the cells of $\mathcal{P}$ such that $C_1, C_2, \dots C_n$ is a path of cells, C_1 and C_n have an edge in common, and $C_i \cap C_j = \emptyset$ for all $j > i + 2$, then $\mathcal{P}$ is called *thin cycle*.

Note that a thin cycle is a polyomino with exactly one hole. In Figure 15 three thin cycles are displayed. In particular, the polyominoes in (A) and (B) have the polyominoes in Figure 12(i)–(iv) as subpolyominoes. This implies that in both cases $\mathcal{M}$ is not a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \{1, \dots, 8\}$. However, the polyomino in (A) is prime, whereas the polyomino in (B) is not. Surprisingly, in the next result we exhibit a class of thin cycles having a prime ideal. The polyomino in Figure 15 (C) belongs to such a class.

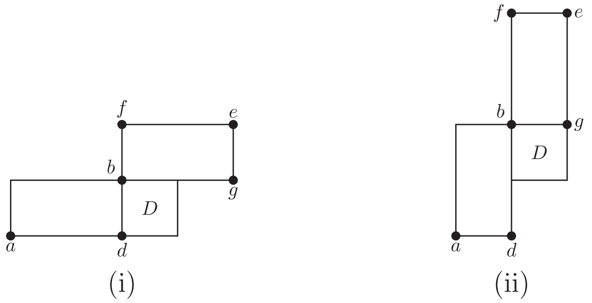


FIGURE 14 Possible positions for the inner intervals $[a, b]$, $[b, e]$, and the cell D, which are contradictory

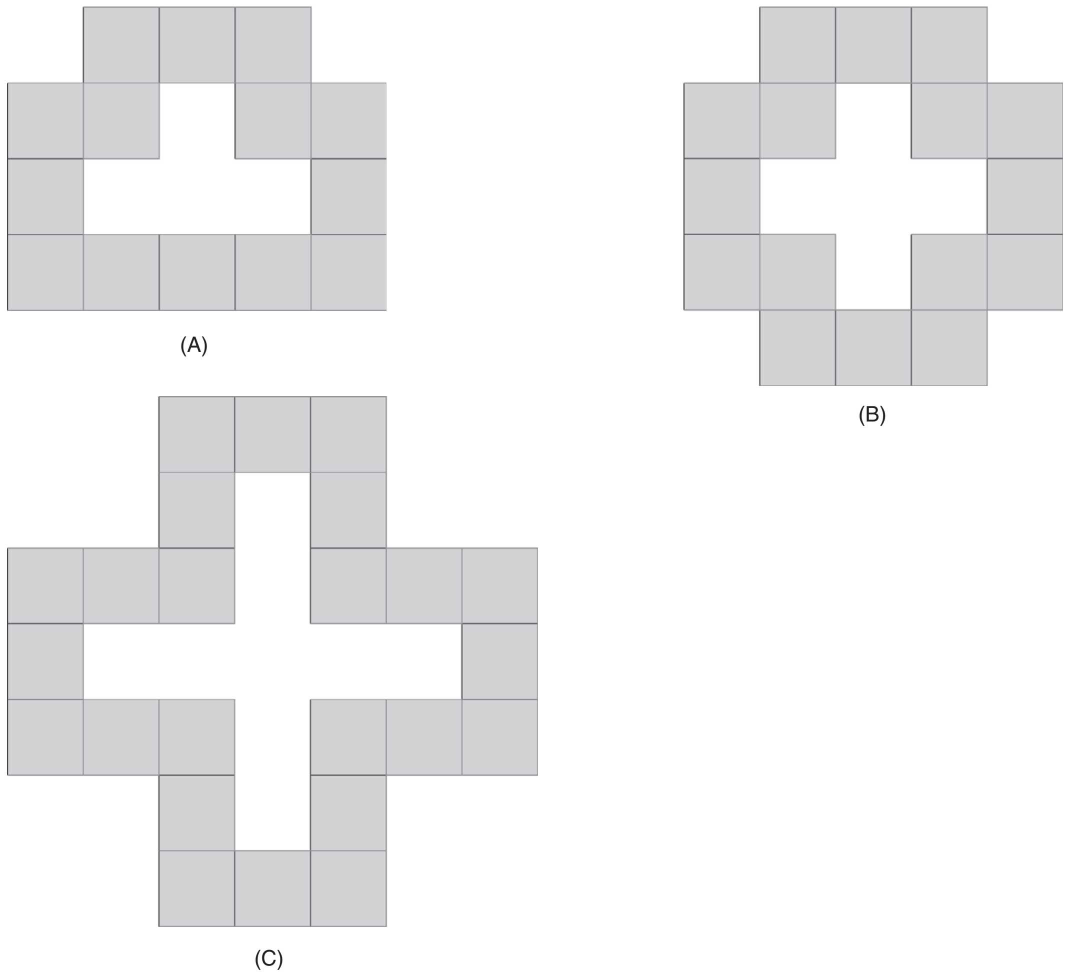


FIGURE 15 Examples of thin cycle polyominoes

Corollary 4.6. *Let $\mathcal{P}$ be a thin cycle polyomino whose all maximal inner intervals have length at least 3. Then $I_{\mathcal{P}}$ is prime.*

Proof. First of all, we observe that such a $\mathcal{P}$ satisfies the condition (2) of Theorem 4.4. In fact, by definition of thin cycle, there are no cells C, D, E and F such that $E, F \in \mathcal{P}$ intersect in one vertex, $C, D \notin \mathcal{P}$ and $C \cap D \cap E \cap F \neq \emptyset$, as in Figure 11. Moreover, by hypothesis, there is no maximal inner intervals of length 2 as in Figure 12. By Theorem 4.4, $\mathcal{M}$ is a quadratic Gröebner basis for $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1, \dots, 8\}$. By Corollary 4.3, the thesis follows. $\square$

As another application of the results obtained for thin polyominoes, we consider the grid polyominoes, that we introduced in [10]. They are prime and, by definition, thin. One can see, by applying Proposition 3.2, that grid polyominoes

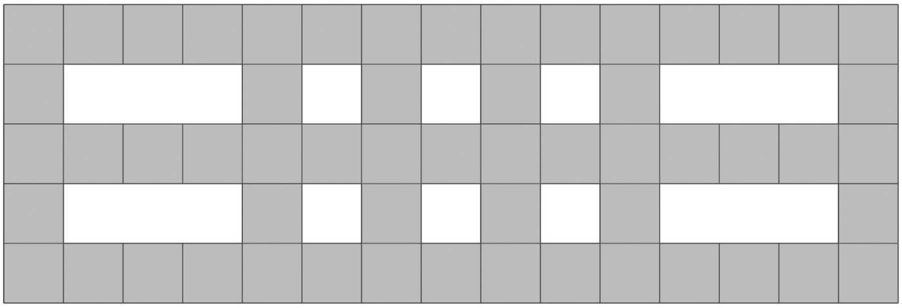


FIGURE 16 A grid polyomino $\mathcal{P}$

have quadratic Gröbner basis with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1, \dots, 8\}$. In the following, we will define a new infinite family of prime polyominoes, obtained by the deletion of certain cells from grid polyominoes. We recall the following definition.

Definition 4.7. Let $\mathcal{P} \subseteq I := [(1,1),(m,n)]$ be a polyomino such that

$$\mathcal{P} = I \setminus \{\mathcal{H}_{ij} : i \in [r],\ j \in [s]\},$$

where $\mathcal{H}_{ij} = [a_{ij}, b_{ij}]$, with $a_{ij} = \left((a_{ij})_1, (a_{ij})_2\right)$, $b_{ij} = \left((b_{ij})_1, (b_{ij})_2\right)$, $1 < (a_{ij})_1 < (b_{ij})_1 < m$, $1 < (a_{ij})_2 < (b_{ij})_2 < n$, and

1. for any $i \in [r]$ and $\ell, k \in [s]$ we have $(a_{i\ell})_1 = (a_{ik})_1$ and $(b_{i\ell})_1 = (b_{ik})_1$;
2. for any $j \in [s]$ and $\ell, k \in [r]$ we have $(a_{\ell j})_2 = (a_{kj})_2$ and $(b_{\ell j})_2 = (b_{kj})_2$;
3. for any $i \in [r-1]$ and $j \in [s-1]$, we have $(a_{i+1j})_1 = (b_{ij})_1 + 1$ and $(a_{ij+1})_2 = (b_{ij})_2 + 1$;
4. $a_{11} = (2,2)$ and $b_{rs} = (m-1, n-1)$.

We call $\mathcal{P}$ a *grid polyomino*.

In Figure 16, an example of grid polyomino is displayed. For more examples, we refer readers to [10].

Remark 4.8. We observe that a grid polyomino $\mathcal{P}$ can be regarded as the disjoint union of two collections of cells, namely $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$, where $\mathcal{P}_1 = \{C \in \mathcal{P} \mid C$ is properly contained in exactly one maximal inner interval of $\mathcal{P}\}$ and $\mathcal{P}_2 = \{C \in \mathcal{P} \mid C$ is properly contained in 2 maximal inner intervals of $\mathcal{P}\}$.

Definition 4.9. Let $\mathcal{P}$ be a grid polyomino with $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ as in Remark 4.8. Let $\mathcal{P}_1'$ be a subset of $\mathcal{P}_1$ such that $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}_1'$ is a polyomino. We call $\mathcal{P}'$ a *subgrid polyomino* of $\mathcal{P}$.

Corollary 4.10. *Let $\mathcal{P}'$ be a subgrid polyomino of a grid polyomino $\mathcal{P}$. Then $I_{\mathcal{P}'}$ is prime.*

Proof. First of all, we claim that $\mathcal{P}'$ satisfies the condition (2) of Theorem 4.4. By contradiction, assume that there exist E and F cells of $\mathcal{P}'$ as in Figure 11, but neither C nor D is a cell of $\mathcal{P}'$. By definition of grid polyomino, either C or D is a cell of $\mathcal{P}$. Without loss of generality, we may assume that C is a cell of $\mathcal{P}$. Then $C \in \mathcal{P}_2$, and $C \notin \mathcal{P}_1$, since C is properly contained in two maximal inner intervals: one containing the cells C and E and the other containing C and F. Then C is still a cell of $\mathcal{P}'$. Moreover, by definition of grid polyomino, the subpolyominoes displayed in Figure 12 are not subpolyominoes of $\mathcal{P}$. Since $\mathcal{P}' \subset \mathcal{P}$, then they are not subpolyominoes of $\mathcal{P}'$ either. By Theorem 4.4, $\mathcal{M}$ is a quadratic Gröebner basis for $I_{\mathcal{P}'}$ with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1, \dots, 8\}$. By Corollary 4.3, the thesis follows. $\square$

In Figure 17, it is shown a subgrid polyomino $\mathcal{P}'$ obtained from the grid polyomino $\mathcal{P}$ displayed in Figure 16 by removing some cells in $\mathcal{P}_1$. By Corollary 4.10, the ideal $I_{\mathcal{P}'}$ is prime.

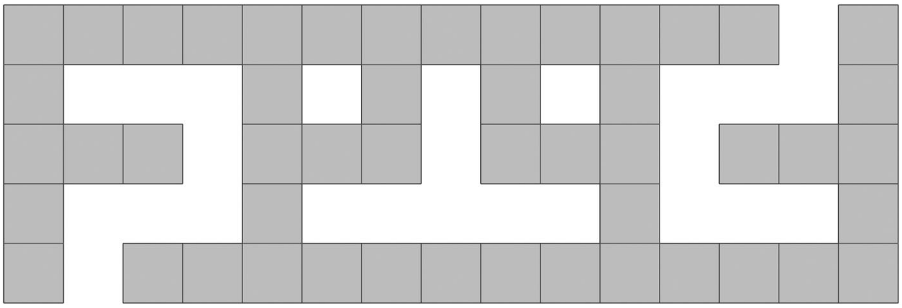


FIGURE 17 A subgrid polyomino of the grid polyomino in Figure 16

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