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APPLICATION OF THE LOGICAL SUSTAINABILITY THEORY TO THE SWEDISH PENSION SYSTEM. LOGICAL SUSTAINABILITY INDICATOR AND BALANCE RATIO, TWO INDICATORS IN COMPARISON

Abstract. Our study focuses on the application of the Logical Sustainability Theory (LST) to the Swedish Pension System, which is a Notional Defined Contribution pension scheme with an automatic balance mechanism that should be able to restore its long-run balance. In our opinion, this mechanism does not ensure the sustainability in a logical-mathematical key. Vice-versa, the LST does it in a logical-mathematical key by means of the Logical Sustainability Indicator (LSI). Indeed, the LST, introduced in Angrisani (2006, 2008) up to Angrisani and Di Palo (2019), is a Theory completely developed for defined contribution pension systems with a funded component. Assumed the Swedish Pension System "adjusted" to the LST hypotheses, the LST basic Beta Indicator, which provides the level of the "unfunded pension liability" in relation to wages, has been calculated. Applying the rule on the rate of return on the pension liability that stabilizes this Indicator, a Sufficient Condition for the sustainability of the pension system with a constant contribution rate is introduced and applied to the Swedish Pension System. Data are provided by the Swedish Pension System Annual Report for year 2017. The comparison between the LSI and the Swedish Balance Ratio is also carried out.

Keywords: Logical Sustainability Theory, Sustainability Indicators, Swedish Pension System, Balance Ratio, Logical Sustainability Indicator.

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1. Introduction

In this paper the Logical Sustainability Theory (LST) is applied to the Swedish Pension System. To this goal, a brief review of the LST, originally introduced in the continuous-time in Angrisani (2006, 2008), is presented as in Angrisani and Di Palo (2019), which developed the Theory in the discrete-time under assumptions of variable mortality and stochastic interest rates.

The LST refers to defined contribution (DC) pension systems with a funded component. In this Theory, the system's assets are a *structural component* of the pension system because they are used to cover a part of the pension liability, and for this reason they are considered as the funded part of the pension liability, i.e., the funded pension liability. As a consequence, the unfunded pension liability, namely the part of the pension liability uncovered by the fund, is also defined.

Starting from the definition of the unfunded pension liability, the key variable of this Theory is also defined. It is referred to as the level of the unfunded contribution rate, and it is in analogy to the Pay-As-You-Go (PAYG) contribution rate used in the PAYG pension systems. By means of the definition of this variable, main conditions that ensure the pension system sustainability are obtained.

These conditions were firstly introduced in Angrisani (2006, 2008) in the continuous-time framework and were applied to pension systems in a stabilization phase in Angrisani and Di Palo (2011, 2012), where a necessary condition provides the extension of Aaron's sustainable rate of return for this type of pension systems.

In our opinion, the LST is the first theory on the sustainability of pension systems not developed under "steady state" hypotheses about demographic-economic variables, hypotheses on which Aaron's theorem (1966) is based. Therefore, in the framework of the LST the sustainability of pension systems is ensured even in the presence of a demographic (and/or economic) wave, see Angrisani and Di Palo (2014) and (2018) p. 2, where "[…] the operating method, based on a general principle, for controlling the demographic wave in the framework of the logical sustainability [...]" is developed in the continuous-time. Although the transition to the corresponding discrete-time model has not been made yet, we believe that it does not present criticalities.

This general principle, named as the Separation Principle, provides that it is necessary to manage by capitalization that component of the pension system that cannot be managed as PAYG.

The Separation Principle is "translated" into the Separation Theorem, and both refer to a pension system in a situation of economic (and/or demographic) stability that is disrupted by a demographic (and/or economic) wave. The main result, provided in Angrisani and Di Palo (2018) p. 2, "[...] consists in proving that in order to face the demographic wave problem … it is not necessary to shift to a fully-funded scheme, despite of the belief of some authoritative authors, see e.g. Feldstein and Ranguelova (2000), Modigliani, Ceprini and Muralidhar (1999) [...]". Specifically, the approach in Angrisani and Di Palo (2018), p. 6, "[...] aims at preserving the stability over time of the pre-existing part of the pension system even if disruptive phenomena of demographic disequilibrium entered the pension system. This means that all that causes disequilibrium, hence active people who outnumber the stability value of the new entrants, should be placed in a separate part of the pension system, which will be financially managed according to the fully-funded scheme.

The two parts of the pension system, both the pre-existing stable part and the part linked to the demographic wave, have to be equivalent under the pension profile. Namely, they have to share the same rules and, in particular, the same

rate of return on the pension liability whereby it is indifferent for an individual whether he/she joins the first or the second subsystem[...].

Furthermore, Angrisani and Di Palo (2018), p. 6, specified that "[...] the first subsystem constitutes the natural prosecution of the already existing stable pension system. Therefore, it continues to receive the same number of new entrants with regular salary dynamics and it remains in a state of economic, financial, and demographic stability. We refer to this as the Pivot Pension System.

Differently, the second subsystem receives the individuals who numerically exceed the stability value of the new entrants and may have unstable salaries or unstable salary dynamics. It has to follow the fully-funded scheme. We refer to this as the Auxiliary Pension System. When the total number of new entrants goes back to the stability value, the Auxiliary Pension System does not receive new participants and becomes a closed group, which exhausts when the last participant dies[...]".

Referring to the "economic wave", this can be identified as the immigrants component in the Swedish context.

Indeed, as shown in the graph of the Orange Report 2019, p. 63, a substantial component of immigrants leaves the Country after a certain number of years.

The work is structured as follows. In Section 2, the LST, as developed in Angrisani and Di Palo (2019), is briefly reviewed, and the new Sufficient Condition for the sustainability of a pension system with a constant contribution rate is introduced in the context of the LST. In Section 3, this new Sufficient Condition is applied to the Swedish Pension System starting from year 2017. Furthermore, the comparison between LST fundamental indicator of sustainability and the Balance Ratio, used to measure the financial position of the Swedish Pension System, is made. Section 4 includes our main conclusions.

2. The Logical Sustainability Theory

The main objective of this Section is to recall the basic definitions of variables and indicators as well as the main conditions for sustainability provided by the LST in the discrete-time and variable mortality and stochastic interest rates framework, as in Angrisani and Di Palo (2019).

2.1. Basics of the Logical Sustainability Theory

Let k, with $k \in N$, denote the time, and let year k, with $k \geq 1$, denote the unitary time interval beginning in *k-1*, excluding *k-1*, and ending in *k*, including *k*, i.e., $(k-1, k]$.

For each time $k \in N$, the following state variables are defined:

1. F_k is the pension system fund, that is the aggregate value of the assets;
2. L_k^A is the pension liability to contributors, referred to also as the latent pension liability, with $L_k^A \geq 0$;
3. L_k^P is the pension liability to retirees, referred to also as the current pension liability, with $L_k^P \geq 0$;
4. L_k^T is the total pension liability, namely $L_k^T = L_k^A + L_k^P$.

For each time $k \in N$, with $k \geq 1$, the following flow (or flow-connected) variables are considered:

1. α_k is the contribution rate, with $\alpha_k \geq 0$;
2. W_k, C_k, and P_k are the wages, the pension contributions, and

the pension disbursements, respectively, with $W_k > 0$, $C_k \geq 0$, and $P_k > 0$, respectively; it is $C_k = \alpha_k W_k$;
3. AL_k^P is the total yearly pension liability that turns, in time k, from latent into current, after the yearly revaluation by rate r_k^{LA}, see following point 5;
4. r_k is the interest rate returned on the fund, F_{k-1}, for year k; it can be described by a stochastic process, see Angrisani et al. (2018);
5. r_k^{LA} is the revaluation rate returned on the pension liability to contributors, L_{k-1}^A, for year k;
6. r_k^{LP} is the revaluation rate returned on the pension liability to retirees, L_{k-1}^P, for year k;
7. $* r_k^{LP}$ is the rate explicitly returned on the pension liability to retirees, L_{k-1}^P, for year k.

Note that all the flow (or flow-connected) variables, with subscript k, are referred to year k, and that both C_k and P_k are paid in arrears.

Furthermore, the evaluation of the state variables, specifically of the fund and the pension liability, is made after that of the flow variables. This means that the evaluation of the fund and the pension liability is made at the end of year k, with $k \geq 1$, after the revenue of the annual contribution, i.e. C_k, the payment of the annual pension expenditure payments, i.e. P_k, and the transfer of the pension liability from contributors to retirees, i.e. AL_k^P.

The dynamics of the fund is connected to contributions and pension expenditure by the basic differential equation

$$F_{k+1} = F_k(1 + r_{k+1}) + C_{k+1} - P_{k+1} \qquad k \in N \tag{1}$$

Equation (1) means that the change in the pension system assets is equal to the return on the assets plus the difference between contributions and pension expenditure.

Another basic evolution equation is given for the total pension liability of the pension system. It is assumed that the rate of return on the latent and current components of the pension liability is the same, that is

$$r_k^{LA} = r_k^{LP} = r_k^L. \tag{2}$$

In a defined contribution pension system, the evolution equation of the pension liability to contributors has to be given by

$$L_{k+1}^{A} = L_{k}^{A}\left(1 + r_{k+1}^{L}\right) + C_{k+1} - AL_{k+1}^{P} \qquad k \in N. \tag{3}$$

As a consequence of equation (3), it follows that the pension liability related to contribution amounts of those who have died during their working years has to be redistributed to other contributors. Possible changes in mortality do not affect the evolution equation for the pension liability to contributors, namely equation (3).

Lastly, we consider the more complex evolution equation of the pension liability to retirees. It has to be taken into account that benefits to retirees earn an implicit return deriving from the progressive extension of life expectancy; with respect to this point, see Angrisani and Di Palo (2019) for the deeper definitions of: a) the rate of the table readjustment, denoted by H_k^T, in year k; b) the rate of the collectivity readjustment after the table readjustment, denoted by H_k^C, in year k. Therefore, the rate of interest that has to be returned to the pension liability to retirees is approximated by

$$r_k^{LP} \approx H_k^T + H_k^C + {}^{*}r_k^{LP},$$

namely by the sum of the two above-mentioned rates of readjustment and the rate explicitly returned on pension liability to retirees for year k.

Therefore, considered the previous observations about the rates, and assumed condition (2), the equation of the evolution of the pension liability to retirees is given by

$$L_{k+1}^{P} = L_{k}^{P}\left(1 + r_{k+1}^{L}\right) - P_{k+1} + AL_{k+1}^{P} \qquad k \in N. \tag{4}$$

Hence, from (3) and (4) the evolution equation for the total pension liability is obtained

$$L_{k+1}^{T} = L_{k}^{T}\left(1 + r_{k+1}^{L}\right) + C_{k+1} - P_{k+1} \qquad k \in N. \tag{5}$$

In addition to the basic evolution equations, in the LST model the following definitions have to be considered.

Let n be the time, with $n \geq 1$.

Definition 1. *A pension system is sustainable in time interval* $[0, n]$ *if and only if the fund, after the contributions revenue and the* *pension benefits payment, is non-negative, i.e.* $F_k \geq 0$ *for each k=0,1,2, ..., n.*

Throughout the paper we assume that

$$0 \leq F_k \leq L_k^T \tag{6}$$

Definition 2. *The unfunded and funded pension liability are defined as*

$$L_k^{UN} = L_k^T - F_k$$
$$L_k^F = F_k,$$

respectively.

For both the above-defined state variables, the evolution equations are provided. Under assumption (2), the evolution equation for the unfunded pension liability is obtained from the difference between equations (5) and (1), and it is given by

$$L_{k+1}^{UN} = L_{k+1}^T - F_{k+1} = L_k^{UN} + L_k^T r_{k+1}^L - F_k r_{k+1}, \qquad k \in N,$$

and also

$$L_{k+1}^{UN} = L_{k+1}^T - F_{k+1} = L_k^T\left(1 + r_{k+1}^L\right) - F_k(1 + r_{k+1}), \; k \in N.$$

In any case, it is worth noting that L_{k+1}^{UN} does not depend on the payment of contributions and pensions in $k + 1$.

Since the funded pension liability coincides with the fund, see Definition 2, it follows that the evolution equation for the funded pension liability is that of the fund, that is equation (1).

Hereinafter, we recall some other basic LST definitions, only those to be used for the aims of this paper. For a complete list of all the definitions, and for a deeper understanding of their meaning, readers can refer to the main paper, see Angrisani and Di Palo (2019).

Definition 3. *The degree of funding of the pension liability is indicated by* Dc_k *and is given by*

$$Dc_k = \frac{F_k}{L_k^T}, \qquad k \in N,$$

with $L_k^T > 0$.

Note that from condition (6) it follows that $0 \leq Dc_k \leq 1$. Hence, a degree of funding of the pension liability equal to one means that the system is fully funded, whereas a degree of funding equal to zero means that the system is zero funded.

Definition 4. *The level of the unfunded pension liability in relation to wages, referred to as the Beta Indicator, is denoted by β_k and it is*

$$\beta_k = \frac{L_k^{UN}}{W_k} = \frac{L_{k-1}^{T}(1+r_k^{L}) - F_{k-1}(1+r_k)}{W_k}, \qquad k = 1, 2, \ldots, n.$$

Definition 5. *The divisor of the provisional total pension liability in the provisional pension liability to retirees is denoted by ν_k and it is*

$$\nu_k = \frac{L_{k-1}^{A}(1+r_k^{LA}) + L_{k-1}^{P}(1+r_k^{LP})}{L_{k-1}^{P}(1+r_k^{LP})}, \qquad k = 1, 2, \ldots, n$$

with $L_{k-1}^{P} \neq 0$.

Note that under assumption (2), it is $\nu_k = \frac{L_{k-1}^{T}}{L_{k-1}^{P}}$ $with\ k = 1, 2, \ldots, n.$

Definition 6. *The divisor of the provisional pension liability to retirees in the pension expenditure is denoted by γ_k and it is*

$$\gamma_k = \frac{L_{k-1}^{P}(1+r_k^{LP})}{P_k}, \qquad k = 1, 2, \ldots, n$$

with $P_k \neq 0$.

Definition 7. *The divisor of the provisional total pension liability in the pension expenditure is denoted by $\gamma_k \nu_k$ and is given by*

$$\gamma_k \nu_k = \frac{L_{k-1}^{T}(1+r_k^{L})}{P_k}, \qquad k = 1, 2, \ldots, n$$

with $P_k \neq 0$.

Note that the definition of variable $\gamma_k \nu_k$ allows the decomposition of the pension expenditure in two components, the unfunded one and the covered one, respectively. Indeed, using Definition 7, the pension expenditure can be expressed as

$$P_k = \frac{L_{k-1}^{T}(1+r_k^{L})}{\gamma_k \nu_k} = \frac{L_{k-1}^{T}(1+r_k^{L}) - F_{k-1}(1+r_k)}{\gamma_k \nu_k} + \frac{F_{k-1}(1+r_k)}{\gamma_k \nu_k}, \; k = 1, 2, \ldots, n \qquad (7)$$

and, if the numerators of both ratios at the right-hand side of (7) are non-negative, the pension expenditure results decomposed in the sum of the following two quantities:

a) the unfunded pension expenditure, expressed by the first term in the sum in (7), namely

$$\frac{L_{k-1}^{T}(1+r_k^{L})-F_{k-1}(1+r_k)}{\gamma_k \nu_k}, \qquad k = 1, 2, \ldots, n;$$

b) the covered pension expenditure, expressed by the second term in the sum in (7), namely

$$\frac{F_{k-1}(1+r_k)}{\gamma_k \nu_k} \qquad k = 1, 2, \ldots, n.$$

Definition 8. *The level of the unfunded contribution rate, or the unfunded contribution rate, is denoted by* α_k^{UN}, *and it is*

$$\alpha_k^{UN} = \frac{\beta_k}{\gamma_k \nu_k}, \qquad k = 1, 2, \ldots, n.$$

From Definition 4, the definition of the Beta Indicator, it follows that

$$\alpha_k^{UN} = \frac{L_{k-1}^{T}(1+r_k^{L})-F_{k-1}(1+r_k)}{\gamma_k \nu_k} \frac{1}{W_k}, \qquad k = 1, 2, \ldots, n,$$

that is α_k^{UN} is the level of the contribution rate that, applied to W_k, makes the corresponding contributions equal to the unfunded pension expenditure.

2.2. Basic conditions and propositions for the pension system sustainability

Let *n* be any time fixed, with $n \in N,\ n \geq 1$. Under assumption (2) the next basic conditions follow.

Theorem 1. The Necessary and Sufficient Condition (NSC) for the pension system sustainability in discrete time interval $[0, n]$
Let the pension system have an initial non-negative fund, i.e. $F_0 \geq 0$.

The pension system is sustainable in $[0, n]$, *with* $n \geq 1$, *i.e.*

$$F_k \geq 0 \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

if and only if the following condition holds

$$-\sum_{h=1}^{k} W_h \left(\alpha_h - \frac{\beta_h}{\gamma_h \nu_h}\right) \prod_{s=1}^{h} \left((1 + r_s)\left(1 - \frac{1}{\gamma_s \nu_s}\right)\right)^{-1} \leq F_0 \ (8)$$

$$for\ each\ k,\ \ k = 1, 2, \dots, n.$$

For the proof see Angrisani and Di Palo (2019).

Remark. Note that ratio $\frac{\beta_h}{\gamma_h \nu_h}$ in (8) is the unfunded contribution rate that, therefore, plays a main role for the pension system sustainability. Indeed, if it results $\alpha_k \geq \alpha_k^{UN}$ for each $k = 1,2, \dots, n$, then the system sustainability is ensured because the quantity at the left hand side in (8) is non-positive and, hence, certainly not greater than the non-negative assets, F_0.

Taking this Remark into account, the following sufficient condition is immediately provided.

The Sufficient Condition (SC) A for the pension system sustainability in discrete time interval $[0, n]$

Let the pension system have an initial non-negative fund, i.e. $F_0 \geq 0$.

The Sufficient Condition for the pension system sustainability in $[0, n]$ *is that contribution rate* α_k *is greater than or equal to the level of unfunded contribution rate* α_k^{UN} *for each* $k = 1,2, \dots, n$, *i.e.*

$$If \qquad \alpha_k \geq \alpha_k^{UN} \qquad for\ each\ k,\ k = 1, 2, \dots, n \qquad (9)$$

$$then \qquad F_k \geq 0 \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

Recalling that $\alpha_k^{UN} = \frac{\beta_k}{\gamma_k \nu_k}$ and that $\beta_k = \frac{L_k^{UN}}{W_k}$, condition (9) can be expressed in an equivalent form by the following condition:

$$\alpha_k \geq \frac{1}{\gamma_k \nu_k} \frac{L_k^{UN}}{W_k}, \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

This form is equivalent to:

$\frac{C_k \gamma_k \nu_k + F_k}{L_k^T} \geq 1, \qquad for\ each\ k, \;\; k = 1, 2, \dots, n$ (10)

Obviously, note that conditions (8) and (9) in the NSC and in SC A, respectively, have to be recursively satisfied.

The quantity at left hand side in (10) is a very important indicator in the LST, see the following definition.

Definition 9. *The Logical Sustainability Indicator (LSI) of the pension system is denoted LSI_k, and is given by*

$$LSI_k = \frac{C_k \gamma_k \nu_k + F_k}{L_k^T}, \quad for\ each\ k, \;\; k = 1, 2, \dots, n.$$

Hence, by means of Definition 9, SC A for the pension system sustainability can be expressed also in the following sufficient condition.

The Sufficient Condition (SC) B for the pension system sustainability in discrete time interval $[0, n]$

Let the pension system have an initial non-negative fund, i.e. $F_0 \geq 0$.

If $\quad LSI_k \geq 1, \qquad for\ each\ k, \;\; k = 1, 2, \dots, n$ (10')

then $\quad F_k \geq 0 \qquad for\ each\ k, \;\; k = 1, 2, \dots, n.$

Obviously, conditions (10') have to be recursively satisfied.

As it results in (9), the unfunded contribution rate plays a fundamental role for the sustainability of the pension system. As the unfunded contribution rate depends on the Beta Indicator, see Definition 8, the stabilization of this Indicator also occupies a determinant role in the matter of the pension system sustainability.

In this regard, the following Proposition 1 is provided.

Proposition 1. The rule for the stabilization of the Beta Indicator in unitary time interval $[k, k+1]$

Let us consider time k, with $k = 0,1,2, \dots n-1$, and assume that

$$0 \le F_k \le L_k^T, \qquad \textit{with} \qquad F_k \ge 0, \quad L_k^T > 0,$$

It is

$$\Delta\beta_{k+1} = 0 \quad \textit{i.e.} \qquad \beta_{k+1} = \beta_k$$

if and only if

$$r_{k+1}^L = \frac{F_k}{L_k^T} r_{k+1} + \frac{L_k^T - F_k}{L_k^T} \sigma_{k+1} = Dc_k r_{k+1} + (1 - Dc_k)\sigma_{k+1}, \quad (11)$$

where r_{k+1} is the interest rate returned on fund F_k in year k+1, and σ_{k+1} is the growth rate of wages in the same year.

We consider a pension system with a constant contribution rate. Using SC A and Proposition 1, i.e. the rule for the stabilization of the Beta Indicator, we obtain the following new proposition for the sustainability of the pension system with constant contribution rate α.

Proposition 2. The Sufficient Condition for the sustainability of a pension system with a constant contribution rate in discrete time interval $[0, n]$
The pension system has constant contribution rate α.
Let the pension system have an initial non-negative fund, i.e. $F_0 \ge 0$.
Let us assume that for each k, with $k = 0, 1, 2 \dots, n$, the rule for the stabilization of the Beta Indicator is recursively applied so that $\beta_k = \beta_0 = \underline{\beta}$, where $\underline{\beta} = \beta_0 = \frac{L_0^T - F_0}{W_0}$, and W_0 is known and greater than 0. Let us set $\underline{\gamma\nu} = \frac{\underline{\beta}}{\alpha}$.
If recursively it results that

$$\gamma_k \nu_k \ge \underline{\gamma\nu}, \ for\ each\ k, \ \ k = 1, 2, \dots, n, \quad (12)$$

then the pension system is sustainable in [0,n], that is

$$F_k \ge 0 \qquad for\ each\ k, \ \ k = 0, 1, 2, \dots, n$$

Proof. By assumption 1), it follows that

$$\frac{1}{\underline{\gamma\nu}} \geq \frac{1}{\gamma_k \nu_k} \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

and also

$$\frac{\underline{\beta}}{\underline{\gamma\nu}} \geq \frac{\underline{\beta}}{\gamma_k \nu_k} \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

or equivalently

$$\alpha \geq \alpha_k^{UN} \qquad for\ each\ k,\ \ k = 1, 2, \dots, n$$

and hence, by SC A, the pension system is sustainable in [0,n], that is

$$F_k \geq 0 \qquad for\ each\ k,\ \ k = 0, 1, 2, \dots, n.$$

Remark. *Note that value* $\underline{\gamma\nu} = \frac{\underline{\beta}}{\alpha}$ *can be interpreted as the minimum value for divisors* $\gamma_k\nu_k, for\ k = 1, 2, \dots n$, *so that SC A for the pension system sustainability holds when the rule for the stabilization of the Beta Indicator is applied, i.e. when* $\beta_k = \beta_0 = \underline{\beta},\ for\ each\ k,\ \ k = 1,2, \dots n$.

3. The application of the LST to the Swedish Pension System and the comparison between the Logical Sustainability Indicator and the Balance Ratio

In this Section, we show how the Sufficient Condition for the sustainability of a pension system with a constant contribution rate, see Proposition 2 p. 8, can be applied to the Swedish Pension System. Furthermore, it is carried out the comparison between the LSI, used to check the sustainability of the pension system in the LST, and the Balance Ratio, used in the Swedish Pension System to measure its "financial position".

It is assumed that the Swedish Pension System is completely "adjusted" to the LST assumptions, in particular, with regard to the evolution equations for the pension liability to contributors, to retirees, and hence to the total population of the pension system, see equations

(3), (4), and (5). With reference to the interest rate of return explicitly recognized to pension liabilities to contributors, r^{LA}, and to retirees, r^{LP}, in the framework of the LST, see appendix A.

3.1. The application of the Sufficient Condition for the sustainability of a pension system with a constant contribution rate to the Swedish Pension System

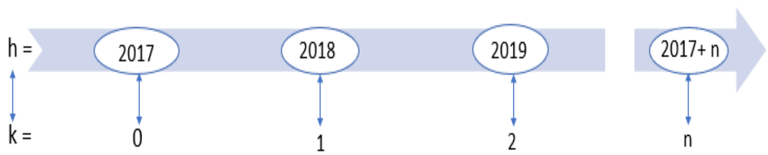
We apply the Sufficient Condition for the sustainability of a pension system with a constant contribution rate, Proposition 2 p. 8, to the Swedish Pension System.

In our exemplification, as established by the LST described in Section 2.1, the state variables are evaluated after the calculation of the flow variables. For the variables related to the Swedish Pension System, the same names of the corresponding variables defined in the LST are used. However, these names are typed in the bold font, when referred to the Swedish Pension System.

In our application we consider $h_0 = 2017$ as the initial calendar year, and hence $[2017, 2017 + n]$, with $n \geq 1$, is the calendar time interval.

To apply the LST to the Swedish Pension System, the following correspondence between the calendar year, denoted by variable h, and the "theoretical" year considered in the LST, denoted by variable k, with $k = h - h_0$, is established, see Figure 1.

Figure 1: Correspondence between calendar year h and "theoretical" year k.



We refer to data of the Swedish Pension System, expressed in millions of SEK, as provided in the Orange Report 2017 (Swedish Pensions Agency, 2018), and reported in the following Table 1.

Table 1. The data of the Swedish Pension System for year 2017.

Year	Total Pension Liability, L_{17}^{T}	Pension System Fund, F_{17}	Pension Contributions, C_{17}	Wages, W_{17}
2017	9,080,454	1,411,896	267,407	1,671,294

Source: Data from Orange Report 2017 (Swedish Pensions Agency, 2018). The value relative to wages is obtained by our own calculation from the corresponding value for pension contributions.

Note that L_{17}^{T}, F_{17} and C_{17} are the total pension liability, the pension system fund, and the pension contributions, respectively, in calendar year 2017 (where the calendar year is indicated by its last two digits)[1].

For the data referred to the total pension liability and the pension system Fund, see p. 10 in the Orange Report 2017 (Swedish Pensions Agency, 2018). The wages, W_{17}, i.e., the income on which the useful contribution for pension purposes is taken, have been "calculated" from the value of the contribution revenue, referred to as Pension Contributions at p.10 in the Orange Report 2017 (Swedish Pensions Agency, 2018), divided by the contribution rate equal to 16 percent[2], see p.20 in the Orange Report 2017 (Swedish Pensions Agency, 2018).

The initial value of the Beta Indicator in calendar year 2017 is:

$$\beta_{17} = \frac{L_{17}^{T} - F_{17}}{W_{17}} = \frac{9{,}080{,}454 - 1{,}411{,}896}{1{,}671{,}294} = 4.5884.$$

We recursively apply in calendar time interval $[2017, 2017 + n]$ the rule for the stabilization of the Beta Indicator so that

$$\beta_h = \beta_{17} = \underline{\beta} \qquad for\ each\ \ h, h \in\ [2017, 2017 + n]$$

Furthermore, we assume that recursively results

$$\gamma_h \nu_h \geq \underline{\gamma\nu} \qquad for\ each\ \ h, h \in\ [2017, 2017 + n]$$

where

[1] This notation is used for all the calendar years in the application of the LST to the Swedish Pension System.

[2] The constant contribution rate is one of the components of the "full package", i.e., the set of instruments, introduced in year 2001 with the reform of the Swedish Pension System to guarantee the generational fairness, see p. 10 in The Swedish Pension System Annual Report 2001, and appendix B of this paper.

$$\underline{\gamma\nu} = \frac{\underline{\beta}}{\alpha} = \frac{\beta_{17}}{\alpha} = \frac{4.5884}{0.16} = 28.68$$

Note that $F_{17} = 1{,}411{,}896$ millions of SEK, hence by Proposition 2, expressed in terms of calendar years, it results that

$$F_h \geq 0 \qquad\qquad for\ each\ h, h \in [2017, 2017 + n],$$

namely the Swedish Pension System is sustainable in time interval $[2017, 2017 + n]$.

Differently, suppose that in year $2017 + n + 1$ the basic assumption of Proposition 2 does not hold any more. Hence, we assume that:

$$\gamma_{17+n+1}\nu_{17+n+1} < \underline{\gamma\nu} = 28.68 \quad \text{or equivalently} \quad \alpha < \alpha^{UN}_{17+n+1}$$

then the thesis of Proposition 2 cannot be deduced, namely it cannot be deduced that

$$F_{17+n+1} \geq 0.$$

Note that Proposition 2 provides only a Sufficient Condition for the sustainability, hence the fund at time $h = 2017 + n + 1$ can also be greater than or equal to zero. In this case, we observe that the NSC for the pension system sustainability must be held in time interval $[2017, 2017 + n + 1]$, see Theorem 1 p. 6.

If we want to restore in year $h = 2017 + n + 1$ the validity of SC A for the pension system sustainability, one possible strategy can be the reduction, in year $h = 2017 + n + 1$, in the rate of return on the pension liability, r^L_{17+n+1}, with respect to the rate of return that stabilizes the Beta Indicator in the same year. In this way, the Beta Indicator takes a value lower than the constant value $\underline{\beta}$ maintained in calendar time interval $[2017, 2017 + n]$.

Therefore, the reduction in the rate of return on the pension liability allows the reduction in the Beta Indicator on a value, β_{17+n+1}, lower than $\underline{\beta} = \beta_{17} = 4.5884$.

In this regard, refer to Proposition 6 in Angrisani and Di Palo (2019). Specifically, in the proof of this Proposition, the relationship between the reduction, in generical year k, in the rate of return and the reduction, in same year k, in the Beta Indicator is quantified, see formula 54 of the cited paper.

Note that the assumption in SC A for the sustainability of a pension system can be satisfied also in the case that the contribution rate is increased, whereas rate of return r^L_{17+n+1} follows rule (11) i.e.

$$r^L_{17+n+1} = Dc_{17+n}r_{17+n+1} + (1 - Dc_{17+n})\sigma_{17+n+1},$$

hence, the level of the Beta Indicator is unchanged.

We can also consider strategies that imply the NSC (Theorem 1) for the pension system sustainability.

A more general study about the "control" of the pension system sustainability in the framework of LST will be carried out in future works.

3.2. The comparison between the Logical Sustainability Indicator of the LST and the Balance Ratio of the Swedish Pension System

We compare, in terms of "effectiveness", the two indicators of sustainability, the LSI defined in the LST, see Section 2, and the Balance Ratio (BR) defined "[…] as a measure that summarizes the financial position of the inkomstpension system […]", see p. 50 in the Orange Report 2019. For a detailed definition of the BR ratio see appendix C.

Note that, in terms of "effectiveness" in guaranteeing the sustainability of the pension system, the LSI certainly ensures the sustainability as it is founded on a logical-mathematical condition, see SC B for the pension system sustainability. The same statement cannot be withdrawn for the BR because there is no logical-mathematical proposition that proves the sustainability of the pension system by means of this indicator.

Indeed, in the LST framework we have that:

If $F_0 \geq 0$ and $LSI_k \geq 1,$ for each $k,\ k = 1, 2, \dots n$

then it follows that

$F_k \geq 0$ for each $k,\ k = 0, 1, 2, \dots, n,$

see Proposition 2.

Differently, in the Swedish Pension System framework we have that:

If $F_0 \geq 0$ and $BR_k \geq 1$, for each k, $k = 1, 2, \ldots, n$

then it does not follow that

$F_k \geq 0$ for each k, $k = 0, 1, 2, \ldots, n$.

This statement is also implied by the fact that the BR, used in the Swedish Pension System, is based on a non-operational variable, the turnover duration, see the definition in Orange Report 2019, reported in appendix C.

4 Conclusions

In this paper, the LST is applied to the Swedish Pension System. In Section 2, we have made a brief review of the LST in the discrete-time framework as in Angrisani and Di Palo (2019). Hence, we have considered the Swedish Pension System "adjusted" to the LST assumptions, in particular, in relation to the evolution equations for the pension liability. In addition, in relation to the pension liability to retirees, we have taken into account the two adjustment rates of the rate of return, r^{LP}, H^T and H^C, as specified in the LST, see appendix A. In the adjustment of the LST to the Swedish Pension System, we have also considered the anticipated rate of interest, i, analogous to the norm of the Swedish Pension System. We have considered the two indicators of the LST, the Beta Indicator and the LSI, and have applied them to the Swedish Pension System.

Finally, the LSI of the LST and the Balance Ratio of the Swedish Pension System are compared. It has been highlighted that the first indicator, LSI, implies the sustainability of the pension system in a logical-mathematical key, whereas the BR does not imply the sustainability of the pension system in a logical-mathematical key.

Furthermore, using the data of the Swedish Pension System for year 2017, value β_{17} of the Beta Indicator has been calculated for this Pension System in this year. As in the Swedish Pension System the contribution rate is constant at 16 percent, we have applied the Sufficient Condition for the sustainability of a pension system with a constant contribution rate, see Proposition 2 p. 8. We have assumed the following relationship between the calendar year, denoted by variable

h, and the "theoretical" year considered in the LST, denoted by variable k with $k = h - h_0$. We have considered the Swedish Pension System in time interval $[2017, 2017 + n]$. We have observed that F_{17} is equal to 1,411,896 millions of SEK. We have supposed that in interval $[2017, 2017 + n]$ the rule for the stabilization of the Beta Indicator is applied, and hence $\beta_h = \beta_{17} = \underline{\beta} = 4.5884$ for each year h, $h \in [2017, 2017 + n]$. We have also supposed that for each year h, in time interval $[2017, 2017 + n]$, condition (12) is recursively verified, i.e. divisor $\gamma_k \nu_k$ is greater than or equal to 28.68, corresponding to the ratio between $\underline{\beta} = 4.5884$ and constant contribution rate $\alpha = 0.16$ of the Swedish Pension System. Hence, by Proposition 2 it follows that the Swedish Pension System is sustainable in time interval $[2017, 2017 + n]$. Differently, we have supposed that in year $2017 + n + 1$ condition (12) is not satisfied, then Proposition 2 cannot be applied. In this case, we have proposed other possible strategies for the sustainability of the Pension System.

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Appendix A. **The interest rate of return of the Swedish Pension System in the LST framework**

The system recognizes the same rate of return on the pension liability to contributors and to retirees. Specifically, the system explicitly recognizes the “full returns” on the pension liability to contributors. With regard to the pension liability to retirees, the explicitly recognized rate of return has to consider both the rates, the rate of the table readjustment, denoted by H_k^T, and the rate of the collectivity readjustment after the table readjustment, denoted by H_k^C, respectively, in year k, so that the rate of return totally recognized on the pension liability to retirees is equal to the rate fully returned on the pension liability to contributors.

In relation to the “adjustment” of the Swedish Pension System to the LST framework, it has to be taken into account that in the calculation of the pension in the Swedish Pension System is already credited to pensions an interest rate in advance of 1.6 percent, see, e.g., the definition of the annuity divisor for the Inkomstpension in Orange Report 2019, p. 95. This rate was initially referred to as norm, see Orange Report 2001, p. 63, where the norm is defined as "…the interest rate of 1.6 percent used when calculating the annuitization divisor and which subsequently is deducted when recalculating pensions with the growth (change) in income (or balance) index…". This method of calculation is applied to make the initial pension higher than it would be without the application of the norm at the time of retirement.

Indeed, in this way "…the divisor is lower than it otherwise would have been, thus raising the value of the initial pension to a level that will be maintained in real terms provided the average income increases by the exact rate of 1.6 percent…", p. 38 in Orange Report 2001.

To consider the interest rate credited to pensions in advance, that is the norm, hereinafter our remark follows.

Remark. *With reference to year* k*, let us denote by* r_k^{LP} *the revaluation rate returned on the pension liability to retirees,* L_{k-1}^{P}*, by* H_k^T *the rate of the table readjustment, by* H_k^C *the rate of the collectivity readjustment after the table readjustment, by* i *the interest rate credited in advance in this case the norm, and by* $*\, r_k^{LP}$ *the rate explicitly returned on the pension liability to retirees,* L_{k-1}^{P}*. Therefore, in year* k *rate* r_k^{LP} *has to face the return already credited in advance, the returns stemming from the readjustment of the table and the collectivity, and also the indexation of the pension liability to retirees at rate* $*\, r_k^{LP}$*, namely it has to be satisfied the following relationship*

$$\left(1+r_k^{LP}\right)=(1+i)\left(1+H_k^T\right)\left(1+H_k^C\right)\left(1+{*}\, r_k^{LP}\right)$$

from which

$$*\, r_k^{LP}=\frac{\left(1+r_k^{LP}\right)}{(1+i)(1+H_k^T)(1+H_k^C)}-1$$

and, hence, in first approximation

$$*\, r_k^{LP}\approx\frac{r_k^{LP}-i-H_k^T-H_k^C}{(1+i)\left(1+H_k^T\right)\left(1+H_k^C\right)} \qquad (13)$$

As an example, let us assume that in year k the revaluation rate returned on the pension liability to retirees, L_{k-1}^{P}, r_k^{LP} is 4%, the rate of the table readjustment, H_k^T, is 0.4%, the rate of the collectivity readjustment after the table readjustment, H_k^C, is 0.1%, and taken into account the norm equal to 1.6%, then $*\,r_k^{LP}$, the rate explicitly returned on the pension liability to retirees, L_{k-1}^{P}, for year k, is approximated by

$$*\,r_k^{LP} \approx \frac{4\% - 0.4\% - 0.1\% - 1.6\%}{(1+1.6\%)(1+0.4\%)(1+0.1\%)} = 1.8$$

Note that, as established in the LST, also for "theoretical" year k=1 (corresponding to calendar year 2018) r_1^L is the rate that must be fully recognized on the pension liability to contributors, whereas $*\,r_1^{LP}$ is the rate that must be recognized on the pension liability to retirees, in order to stabilize the Beta Indicator. Therefore, in "theoretical" year k=1 (corresponding to calendar year 2018) rate of return $*\,r_1^{LP}$ is provided, in first order of approximation, by the following formula (14)

$$*\,r_1^{LP} \approx \frac{r_1^{LP} - i - H_1^T - H_1^C}{(1+i)(1+H_1^T)(1+H_1^C)} \qquad (14)$$

where for "theoretical" year k=1 (corresponding to calendar year 2018) it is:

- $*\,r_1^{LP}$ is the rate explicitly returned on the pension liability to retirees;
- r_1^{LP} is the totally revaluation rate returned on the pension liability to retirees;
- r_1^{LA} is the revaluation rate returned on the pension liability to contributors;
- H_1^T is the rate of the table readjustment;
- H_1^C the rate of the collectivity readjustment after the table readjustment;
- i is the "norm" equal to 1.6%;

Furthermore, in according to assumption (2) in Section 2, we have $r_1^{LP} = r_1^{LA} = r_1^L$ referred to "theoretical" year $k = 1$ (corresponding to calendar year 2018).

Appendix B. Overview of the Swedish Pension System

Since 2001, Sweden has reformed the pension system by migrating from the PAYG-DB scheme, with an additional component linked to a supplementary benefit dependent on income, called ABT, towards a mixed system characterized by a notional PAYG-DC component and a fully funded called Inkomstpension and Premium Pension respectively (Palmer, 2000).

In the presentation of the reform, contained in the first Swedish Pension System Annual Report 2001, there is great enthusiasm about the structure given to the reform based on principles of fairness between generation. This vital issue, it is pursued through the "full package" which is composed of:

- fixed contribution rate
- average income as a basis for calculating indexation
- adjustment of pension expenditure based on the change in average life before 65 years of age
- buffer fund
- automatic balancing mechanism.

A key point with respect to the past is although the setting remains actuarial, the analysis of the system is performed without projections. The valuation of assets and liabilities is conducted only on the basis of what is observable. The adoption of a contribution-based benefit calculation system determines a performance based on the trend of demographic, economic and financial elements which is reported through an annual report by the National Social Insurance Board. The possibility of taking advantage of detailed periodic reporting has made the Swedish system a benchmark among mixed pension systems. The periodic report is done according to two principles: the first of the clarity according to which the report must provide information on all aspects that influence the financial position and the value of pensions, while the second refers to the setting of an accounting scheme as close as possible to the private insurance company.

The new system allows the contribution rate of 18.5% to be split into 16% for the component PAYG and the remaining 2.5% for the fully funded component. See Swedish Pension System Annual Report 2001.

Appendix C. Overview of the Balance Ratio of the Swedish Pension System

In its first formulation, denoted by k the calendar year if the variable refers to flows, end of the calendar year if the variable refers to stocks, the balance ratio (BR) is calculated as

$$BR_k = \frac{Assets}{Liabilities} = \frac{CA_k + F_k}{L_k^T}$$

where CA_k is the Contribution Asset and F_k the Buffer Fund.

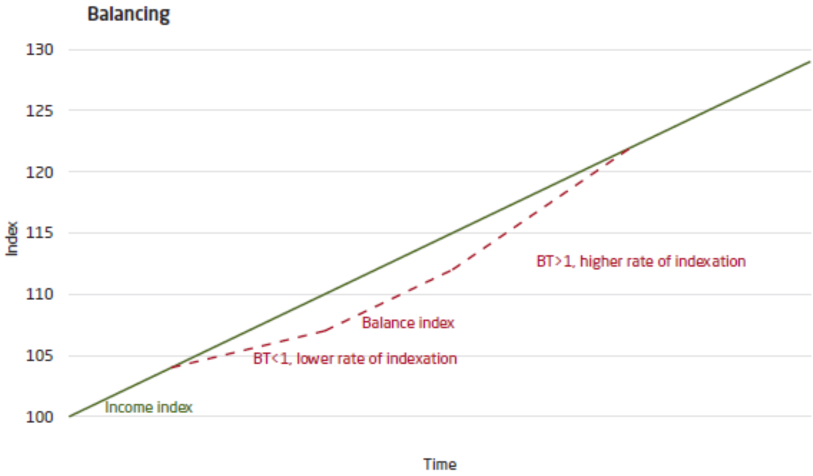
The Contribution Asset is obtained from:

$$CA_k = TD_k \cdot C_k$$

where TD_k is the Turnover Duration at time k,"*[...] the expected time elapsing from pension credit has been earned until the pension is paid out in the pay-as-you-go system, measured as an average that is weighted for pension credits and pension amounts",* see p.64 in Swedish Pension System Annual Report 2001.

C_k are the contributions obtained at time k from the product of the wages at time k, W_k, to contribution rate α, with $\alpha = 0.16$.

If the balance ratio is greater than or equal to one the system is sustainable while if it is less than 1 the system is not sustainable, and it is necessary to activate the balance mechanism which reduces the rate of return to be recognized for the members of the system.



BT Balance ratio

Source: Orange Report: Annual report of the Swedish Pension System 2019.