Optimal Cantilever Dynamic Vibration Absorbers by Timoshenko Beam Theory

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Optimal cantilever dynamic vibration absorbers by Timoshenko Beam Theory

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Professor Bruno Piombo has shown us that the passion for the research and for teaching is a neverending gift.

Abstract. A double–ended cantilever beam as a distributed parameter dynamic vibration absorber has been applied to a single–degree–of–freedom system subjected to harmonic forces.
In this investigation, the beam has been analyzed under the well known model of Timoshenko and the computation of best parameters is based on the Chebyshev's optimality criterion.
This is somewhat novel in the field since:

- the design of cantilever beams as dynamic vibration absorbers is usually made under the hypotheses of the Euler–Bernoulli theory;
- it is the first time that the Chebyshev's criterion is applied to the design of such devices.

For a ready use of the results herein presented, design charts allow a quick choice of optimal parameters such as tuning ratio and mass ratio.

1 Introduction

The classical dynamic vibrations absorber is made up of two masses. The first one is subjected to an harmonic load which lead to a vibrational motion of this mass, the second mass is connected to the main mass by means of an spring element. Thus choosing properly the weight of the second mass and the spring stiffness, the vibration amplitude of the main mass could be reduced and, under ideal conditions, also cancelled [4, 12].

In fact when an absorbing mass-spring system is attached to the main mass and the resonance of the absorber is tuned to match that of the main mass, the motion of the main mass is reduced to zero at its resonance frequency (Figure 1).

The test case proposed in this paper concerns a double–ended cantilever beam used as a dynamic vibration absorber [8]. As a consequence the parameters to be set in order to reduce the vibrations of the main mass are the intrinsic elasticity of the beam and its weight. For a more faithful modeling of the beam behavior, the authors have deduced the dynamic equations of the system under analyisis by means of the Timoshenko's model [7]. These equations have been used under the conditions set by the Chebyshev's theorem in order to define the optimal features of the beam (*e.g.* cross section area, length, thickness) [7]. The results have been compared with the ones obtained following the Euler–Bernoulli's model of the beam as reported by Jacquot and Foster [9].

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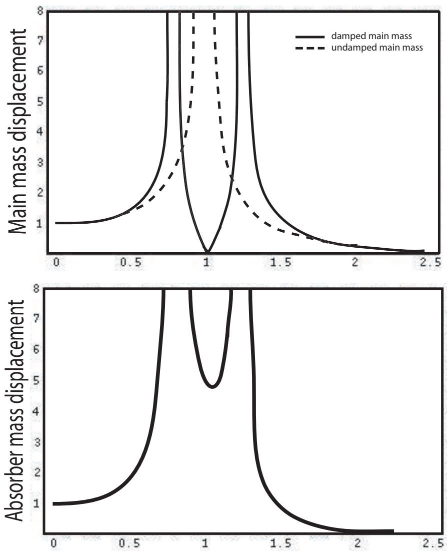

Figure 1: Plots of the mass displacements

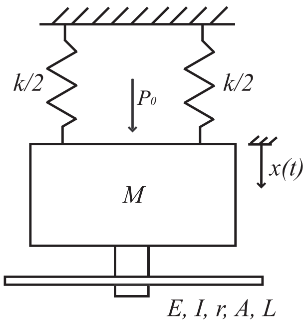

Figure 2: Scheme of the dynamic vibrations absorber

2 System modeling

The undamped system, shown in Figure 2, is composed of a spring–supported lumped mass which is free to move only vertically.

Attached to the mass there is the double–ended cantilever beam as shown. Separating the system into three parts as shown in Figure 3, the method of superposition is applied to examine the coupling between the subsystems.

The equation of motion for the main mass M is

$$M\frac{\delta^2 w}{\delta t^2} + Kw = P_0 e^{j\omega t} + 2V_0 e^{j\omega t}\ , \tag{1}$$

where the $V_0 e^{j\omega t}$ term is the yet unknown vibration absorber force for a single beam on the main mass. This force can be obtained as the shear force at the root of a displacement–excited cantilever beam. The governing equation for the cantilever beam, under the assumption of Timoshenko bending theory, is

$$EI_n\frac{\delta^4 y}{\delta x^4} + A\rho\frac{\delta^2 y}{\delta t^2} - \rho I_n\left(1 + \frac{E}{\chi G}\right)\frac{\delta^4 y}{\delta x^2 \delta t^2} + \frac{\rho^2 I_n}{\chi G}\frac{\delta^4 y}{\delta t^4} = 0 \tag{2}$$

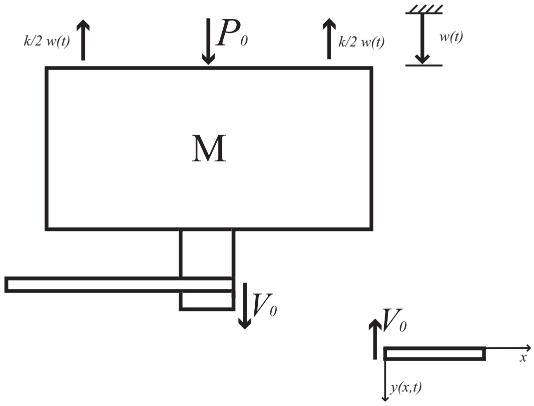

Figure 3: Freebody diagram of the system

with the following boundary conditions:

$$y\,(0,t) = W_0 e^{j\omega t} \tag{3a}$$

$$\varphi\,(0,t) = 0 \tag{3b}$$

$$\frac{\delta\varphi\,(L,t)}{\delta x} = 0 \tag{3c}$$

$$\frac{\delta y\,(L,t)}{\delta x} - \varphi(L,t) = 0 \tag{3d}$$

In order to obtain a steady–state solution to Eq. (2), we assume a solution of the form

$$Y\,(x,t) = Y\,(x)\,\mathrm{e}^{j\omega t} \tag{4}$$

which will yield the spatial complex amplitude distribution $Y\,(x)$. The shear force amplitude at the root of the cantilever is then

$$V\,(x,t) = \chi AG\left(\psi - \frac{\delta y}{\delta x}\right)\ . \tag{5}$$

The Eq. (2), subjected to boundary conditions (3), using expression (5) gives a shear force amplitude of

$$V_0 = (U_1 + U_2)\,W_0 \tag{6}$$

with

$$U_1 = \frac{AT^4\left[\cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1 T^2 - \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2 T^2\right]\left(\alpha_1^2+\alpha_2^2\right) ro^3\lambda^6\Omega_1^6}{\left(\alpha_2^2\alpha_1^2\chi^2 G^2 Q\right)}$$

where $B_1 = \left(\alpha_1^2+\alpha_2^2\right)\rho^3\lambda^6\Omega_1^6$,
and

$$U_2 = \frac{A\rho^2 T^4\left[-\cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1^3 - \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2^3\right]\left(\alpha_1^2+\alpha_2^2\right)\lambda^4\Omega_1^4}{\left(\alpha_1^2\alpha_2^2\chi GQ\right)}$$

where $B_2 = \left(\alpha_1^2+\alpha_2^2\right)\rho^4\Omega_1^4$
and

$$Q = Q_1 + Q_2 + Q_3 \tag{7}$$

$$\begin{aligned} Q_1 = &- \frac{T^6\Omega_1^6\lambda^6\rho^3}{\left(\chi^3 G^3 \alpha_1^2\alpha_2^2\right)}[-\alpha_2^2 \sin(\alpha_1 L)\sinh(\alpha_2 L) - 2\alpha_1\alpha_2 \\ &+ \alpha_1^2 \sin(\alpha_1 L)\sinh(\alpha_2 L) + 2\alpha_1\alpha_2\cos(\alpha_1 L)\cosh(\alpha_2 L)] \end{aligned} \tag{8}$$

and

$$\begin{aligned} Q_2 = &\frac{T^4\Omega_1^4\lambda^4\rho^2}{\left(\alpha_1^2\alpha_2^2\chi^2G^2\right)}[2\alpha_1\alpha_2^3\cos(\alpha_1 L)\cosh(\alpha_2 L) + 2\alpha_1^3\alpha_2 \\ &- 2\alpha_1\alpha_2^3 - \alpha_1^4\sin(\alpha_1 L)\sinh(\alpha_2 L) - 2\alpha_1^3\alpha_2\cos(\alpha_1 L)\cosh(\alpha_2 L) \\ &+ 2\alpha_1^2\alpha_2^2\sin(\alpha_1 L)\sinh(\alpha_2 L) - \alpha_2^4\sin(\alpha_1 L)\sinh(\alpha_2 L)] \end{aligned} \tag{9}$$

and

$$\begin{aligned} Q_3 = &- \frac{\rho T^2\Omega_1^2\lambda^2}{\alpha_1^2\alpha_2^2\chi G}[\alpha_1\alpha_2^5\cos(\alpha_1 L)\cosh(\alpha_2 L) \\ &- \sin(\alpha_1 L)\,\alpha_1^4\alpha_2^2\sinh(\alpha_2 L) + \alpha_1^2\sin(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_2^4 \\ &+ \alpha_2\alpha_1^5\cos(\alpha_1 L)\cosh(\alpha_2 L) + 2\alpha_1^3\alpha_2^3]\ . \end{aligned} \tag{10}$$

The steady–state solution to (1) is

$$w(x,t) = W_0 e^{j\omega t} \tag{11}$$

where W_0 is the complex amplitude, and the following equation must hold

$$\left(-\lambda^2 + 1\right) W_0 - \frac{P_0}{k} - \frac{2V_0}{k} = 0\ . \tag{12}$$

Note that the second forcing term on the right side of (11) is a function of the complex vibratory amplitude of the mass W_0. Solving equation (12) for complex amplitude W_0 as a function of the external forcing function amplitude P_0 one obtains

$$W_0 = \frac{C}{D} \tag{13}$$

and

$$C = -\frac{P_0\mu\alpha_1^2\alpha_2^2\chi^2G^2Q}{2} \tag{14}$$

where

$$\begin{aligned} D = &\rho A\Omega_1^2[\left(L\alpha_2^2Q\alpha_1^2\lambda^2 - \alpha_2^2Q\alpha_1^2L\right)G^2\chi^2 + \\ &[-\cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1^5 - \cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1^3\alpha_2^2 \\ &- \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2^3\alpha_1^2 - \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2^5]T^4\Omega_1^2\mu\rho\lambda^4G\chi \\ &+ [\cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1^3 + \cos(\alpha_1 L)\sinh(\alpha_2 L)\,\alpha_1\alpha_2^2 + \\ &- \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2\alpha_1^2 - \sin(\alpha_1 L)\cosh(\alpha_2 L)\,\alpha_2^3]T^6\Omega_1^4\mu\rho^2\lambda^6] \end{aligned} \tag{15}$$

In order to get maximum benefit from the present analysis some nondimensional quantities are introduced. The tuning ratio T is the ratio of the first natural frequency of the cantilever to the natural frequency of the main lumped parameter system

$$T = \frac{\omega_a}{\Omega_1} \tag{16}$$

The mass ratio μ is the ratio of the total absorber mass to that of mass M

$$\mu = \frac{2\rho AL}{M} \tag{17}$$

The frequency ratio λ is the ratio of the frequency to the natural frequency of the $k - M$ combination of

$$\lambda = \frac{\omega}{\Omega_1} \tag{18}$$

The static deflection of the main system is defined to be

$$W_{st} = \frac{P_0}{k} \tag{19}$$

The dimensionless frequency responce function is then

$$\gamma = \left| \frac{W_0}{W_{st}} \right| = \left| \frac{F}{H} \right| \tag{20}$$

where

$$F = \alpha_1^2 \alpha_2^2 \chi^2 G^2 QL \tag{21}$$

and

$$\begin{aligned} H = {} & \left(L\alpha_2^2 Q\alpha_1^2 \lambda^2 - \alpha_2^2 Q \alpha_1^2 L \right) G^2 \chi^2 + T^4 \Omega_1^2 \mu \rho \lambda^4 G \chi \\ & [- \cos(\alpha_1 L) \sinh(\alpha_2 L) \alpha_1^5 - \cos(\alpha_1 L) \sinh(\alpha_2 L) \alpha_1^3 \alpha_2^2 \\ & - \sin(\alpha_1 L) \cosh(\alpha_2 L) \alpha_2^3 \alpha_1^2 - \sin(\alpha_1 L) \cosh(\alpha_2 L) \alpha_2^5] \\ & + T^6 \Omega_1^4 \mu \rho^2 \lambda^6 [\cos(\alpha_1 L) \sinh(\alpha_2 L) \alpha_1^3 + \cos(\alpha_1 L) \sinh(\alpha_2 L) \alpha_1 \alpha_2^2 \\ & - \sin(\alpha_1 L) \cosh(\alpha_2 L) \alpha_2 \alpha_1^2 - \sin(\alpha_1 L) \cosh(\alpha_2 L) \alpha_2^3] \end{aligned} \tag{22}$$

with

$$\alpha_1 = \frac{1}{2} \lambda T \Omega_1 \sqrt{2} \sqrt{\frac{\rho}{E}} \sqrt{\frac{E}{\chi G} + 1 + \sqrt{\left(\frac{E}{\chi G} - 1 \right)^2 + \frac{4AE}{\rho \lambda^2 T^2 \Omega_1^2 I_n}}} \tag{23}$$

and

$$\alpha_2 = \frac{1}{2} \lambda T \Omega_1 \sqrt{2} \sqrt{\frac{\rho}{E}} \sqrt{-\frac{E}{\chi G} + 1 + \sqrt{\left(\frac{E}{\chi G} - 1 \right)^2 + \frac{4AE}{\rho \lambda^2 T^2 \Omega_1^2 I_n}}} \tag{24}$$

3 Chebyshev's theorem

In this section are recalled the main proposition of the Chebychev's theorem for the search of optimal parameters [9]. Let $f(x)$ be a continuous function in $[a, b]$ and $p(x)$ an approaching polynomial belonging to the class P_n of polynomials with degree less or equal to n. As specified by Chebyshev's theorem, the best uniform approximation is attained when the condition

$$\min \quad \max \quad |f(x) - p(x)| \tag{25}$$

is fulfilled. The solution to the minimization problem stated by (1) is unique and it can be found considering the following theorem: Let $f(x)$ be a continuous function in $[a, b]$ and $p(x)$ the best uniform approaching polynomial of degree n. Moreover, let

$$E_n = \max \quad |f(x) - p(x)| \tag{26}$$

and

$$\epsilon(x) = f(x) - p(x). \tag{27}$$

There are at least $(n+2)$ points $a \le x_1 < x_2 \ldots < x_{n+2} \le b$ where $\epsilon(x)$ assumes the values $\pm E_n$ and with alternating signs:

$$\epsilon(x_i) = \pm E_n \tag{28}$$

with $i = 1, 2, \ldots, n+2$ and

$$\epsilon(x_i) = -\epsilon(x_{i+1}) \tag{29}$$

with $i = 1, 2, \ldots, n+1$. Hence the best uniform approaching function is completely characterized by the property of equioscillation at $(n+2)$ points. This property is the basis of numerical schemes for computing the approximant polynomial. The Chebyshev's theorem allows us to determine the optimal values of μ and T such that the curve γ, versus λ, has two peak values with minimum distance from a straight line L_1, where L_1 is initially unknown. The following system of non–linear algebraic equations can be written

$$\left(\frac{\delta\gamma}{\delta\lambda}\right)_{\lambda=\lambda_1} = 0 \tag{30a}$$

$$\left(\frac{\delta\gamma}{\delta\lambda}\right)_{\lambda=\lambda_2} = 0 \tag{30b}$$

$$\left(\frac{\delta\gamma}{\delta\lambda}\right)_{\lambda=\lambda_3} = 0 \tag{30c}$$

$$-\gamma(\lambda_1) + L_1 + \Delta = 0 \tag{30d}$$

$$-\gamma(\lambda_2) + L_1 - \Delta = 0 \tag{30e}$$

$$-\gamma(\lambda_3) + L_1 + \Delta = 0 \tag{30f}$$

where Δ is the maximum deviation of the responce curve from the value L_1. A curve attains a maximum or a minimum at frequency ratios λ_1, λ_2 and λ_3. Therefore, system (30) is composed of six equations with seven unknown variables λ_1, λ_2, λ_3, μ, T, L_1 and Δ. Solving the system of non–linear equation for different and prescribed values of μ, it can be computed the numerical values of the optimal parameters.

4 Numerical example

Considering a system where the lenght of the beam is kept constant, the optimized frequency response function of the main mass is obtained making use of the design charts presented in Figures 4 and 5. In particular, in Figure 4 is shown the graph which relates the optimal values of mass ratio as a function of the main mass M. In Figure 5 is reported the graph of the optimal tuning ratio T as a function of μ_{opt}. Thus once known the value of μ_{opt} from the previous step, is possible to obtain also the optimal value of tuning ratio T. Using the equations (20) with both of the optimal values previously obtained, it is possible to define the final behaviour of the frequency response function of the main mass.

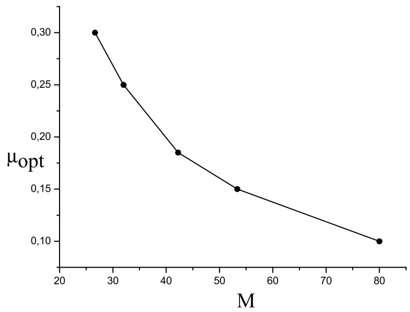

Figure 4: Optimal values of mass ratio

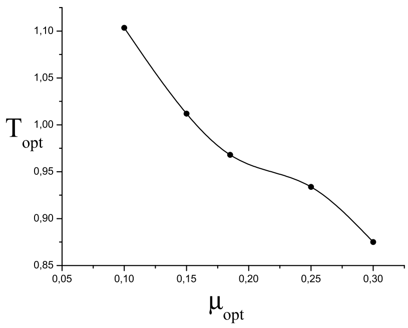

Figure 5: Optimal tuning ratio

The procedure previously discussed has been applied to a main mass of a primary system whose vibration needs to be controlled. The value of main mass is $M = 42.243$ Kg and the vibration absorber is a uniform beam with the following pertinent specifications: $L = 1$ m, $A = 5 \cdot 10^{-4}$ m^2, $E = 2 \cdot 10^{11}$ N/m^2, $G = 0.808 \cdot 10^{11}$ N/m^2, $\rho = 8000$ Kg/m^3, $I_n = 10^{-9}$ m^4 and $\chi = 0.833$. The optimal solution for the Timoshenko beam theory is $T_{opt} = 0.968$ and $\mu_{opt} = 0.185$.

In conclusion, the main advantages in using the present technique are:

- the two peacks of the main mass maximum displacement value are levelled;

- the use of Timoshenko beam theory guarantee a more faithful modeling of the beam dynamic behavior.

The optimal parameters, using the Euler–Bernoulli beam theory, are $T_{opt} = 0.865$ and $\mu_{opt} = 0.2$. These are not very far from the optimal solution obtained by means of Timoshenko theory. Figure 6 refers to a comparison between Euler–Bernoulli beam theory and Timoshenko beam theory with the computed optimal parameters $T_{opt} = 0.968$ and $\mu_{opt} = 0.185$.

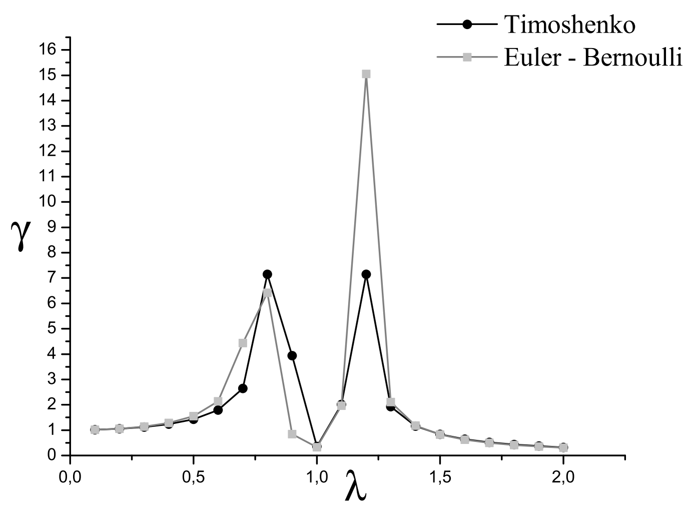

Figure 6: Comparison between Timoshenko and Euler-Bernoulli theory

5 Conclusions

The authors have proposed a new method for the optimal design of a double–ended cantilever beam as a dynamic vibration absorber for a lumped–parameter single degree–of–freedom vibration system. The governing equation for the cantilever beam was developed under the assumption of Timoshenko bending theory. The Chebyshev's criterion was applied and the design charts for optimal beam–type absorber prepared. The results should be useful to designers of machine elements and structural systems.

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NOMENCLATURE

A	area
E	Young's modulus
G	shear modulus
I_n	area moment of inertia
k	stiffness coefficient
L	lenght
L_1	straight line
M	main mass
P_0	external forcing function amplitude
Q	solving equation of the system
$\dot{q}$, $\dot{v}$	generalized velocities
T	tuning ratio
V	shear force
W_0	complex amplitude
W_{st}	static deflection of the main system
Y	spatial complex amplitude distribution
λ	frequency ratio
μ	mass ratio
ρ	mass density
χ	shear factor
Ω_1	natural frequency of the k–M combination
ω	input frequency
ω_a	first natural frequency of the cantilever